Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on AdS_3

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1 Introduction

The purpose of this paper is to prove the following result, which was announced in [6] :

Theorem 1.1. *Let (M, g) be a three-dimensional maximal globally hyperbolic spacetime, locally modelled on the anti-de Sitter space AdS_3, with closed orientable Cauchy surfaces. Then, M admits a CMC time function τ. Moreover, the function τ is unique and real-analytic, and every CMC spacelike compact surface in M is a fiber of τ.*

Theorem 1.1 deals with three-dimensional spacetimes whose sectional curvature is constant and negative. We used the equivalent formulation "locally modelled on the anti-de Sitter space AdS_3" to emphasize the fact that the geometry of AdS_3 and the $(O(2, 2), AdS_3)$-structure of the spacetime will play a crucial role in our proof of the Theorem 1.1 (see section 3).

We recall that a spacetime (M, g) is said to be *globally hyperbolic* if there exists a spacelike hypersurface Σ in M such that every inextendable non-spacelike curve intersects Σ at one and only one point. Such an hypersurface Σ is called a *Cauchy surface*. A globally hyperbolic spacetime (M, g) locally modelled on AdS_3 is said to be *maximal* if any embedding of M in a globally hyperbolic spacetime locally modelled on AdS_3 is surjective. Notice that, if a spacetime (M, g) admits a closed Cauchy hypersurface, then every Cauchy surface in M is closed, and every closed spacelike hypersurface in M is a Cauchy hypersurface. Moreover, it follows from Mess' work ([15]) that a spacetime locally modeled on AdS_3 is maximal globally hyperbolic with compact Cauchy surfaces if and only if it is maximal with respect to the property that there is a closed spacelike surface through every point.

A *time function* on a spacetime (M, g) is a submersion $\tau : M \to \mathbb{R}$ such that τ is strictly increasing along every future-directed timelike curve. Every globally hyperbolic spacetime admits (many) time functions. Conversely, a spacetime admitting a time function which is surjective when restricted to any inextendable causal curve is globally hyperbolic; in this case, the level sets of τ are Cauchy hypersurfaces.

A *CMC time function* on a spacetime (M, g) is a time function $\tau : M \to \mathbb{R}$ such that, for every $\theta \in \mathbb{R}$, the set $\tau^{-1}(\theta)$ is a spacelike hypersurface with constant mean curvature θ. In particular, a spacetime which admits a CMC time function is foliated by spacelike hypersurfaces with constant mean curvature. The foliation defined by a CMC time function is sometimes called a *York slicing*.

Before discussing the implications of Theorem 1.1, let us say that there exist analogs of this theorem for spacetimes with constant non-negative curvature (see [1]and [5] for the flat case in any dimension, and [7] for the positive curvature case in dimension 3). In fact, three-dimensional maximal globally hyperbolic spacetimes with constant curvature and compact Cauchy surfaces always admits a CMC time function, except for three special types of spacetimes : up to finite coverings, these exceptionnal spacetimes are quotients of the Minkowski space Min_3 by a group of spacelike translations, quotients of certain domains of the de Sitter space dS_3 by rank 2 abelian groups of parabolic isometries, and the de Sitter space dS_3 itself. Even in these special cases, there is a foliation by compact closed CMC surfaces, which is unique except in the case of the de Sitter space itself.

The major motivation for proving Theorem 1.1 comes from the links of this theorem with the (vacuum) Einstein equation.

First of all, let us recall that, in dimension 3, the vacuum Einstein equation (with cosmological constant) reduces to the requirement that the curvature of the spacetime is constant. In particular, the solutions of the three-dimensional vacuum Einstein with negative cosmological constant are exactly the spacetimes with negative constant curvature.

The notion of global hyperbolicity is linked with the most usual way to find solutions of the Einstein equation: to solve the associated Cauchy problem. This approach, in dimension $2+1$, consists in considering a surface Σ with a Riemannian metric $\bar{g}$ and a symetric 2-tensor II, and trying to find a Lorentzian metric g on $M = \Sigma \times]-1,+1[$, such that g satisfies the Einstein equation, such that $\bar{g}$ is the restriction of g on $\Sigma = \Sigma \times \{0\}$ and such that II represents the second fundamental form of $\Sigma = \Sigma \times \{0\}$ in $M = \Sigma \times]-1,+1[$. For the problem to admit a solution, the initial data $(\Sigma, \bar{g}, II)$ must satisfy the *constraint equations* (for geometers, the Gauss-Codazzi equations). Conversely, Choquet-Bruhat theorem ([9]) states that every initial data satisfying the constraint equation leads to a solution, which, by the nature of the process, is globally hyperbolic. Moreover, according to Choquet-Bruhat and Geroch ([10]), there is a unique maximal globally hyperbolic solution (up to isometry).

The main difficulty when dealing with the Cauchy problem is the invariance of Einstein equation under the action of diffeomorphisms, leading to an infinite dimensional space of local solutions. To bypass this difficulty, one has to choose a gauge, *i.e.* to reduce the dimension of the space of solution by imposing additional constraints. The method used by Choquet-Bruhat consists in considering local coordinates (x_1, x_2, x_3), such that the surface Σ corresponds to $x_3 = 0$, and to demand (with no loss of generality) the harmocity of these coordinates with respect to the (unknown) Lorentzian metric g. In such coordinates, the Einstein equation becomes a quasi-linear hyperbolic equation for which classical techniques apply.

Another similar method is to restrict to the case where each spacelike surface $\Sigma \times \{*\}$ is a CMC surface. Then, the equation simplifies dramatically. The main drawback of this approach is that one has to assume the existence of a CMC surface. Our theorem shows that this assumption, which is *a priori* very restrictive, is automatically fulfilled for the three-dimensional vacuum Einstein equation with negative cosmological constant. Hence, the remarkable simplification of the Einstein equation described above, that one could call "CMC reduction", applies in full generality.

The CMC reduction is the essential tool of the reduction described by V. Moncrief of Einstein equation to a non-autonomous Hamiltonian flow (that we call *Moncrief flow*) on the cotangent bundle of the Teichmüller space of Σ ([16]). Moncrief flow can be described as follows : for every trajectory $\gamma : \mathbb{R} \to T^*\mathrm{Teich}(\Sigma)$, there exists a maximal globally hyperbolic space M with CMC time function τ such that the projection of $\gamma(t)$ on $\mathrm{Teich}(\Sigma)$ is the conformal class $[\bar{g}_t]$ of the Riemannian metric of the surface $\Sigma_t = \tau^{-1}(t)$, and the cotangent vector $\gamma(t)$ is a holomorphic quadratic form extracted from the divergenceless and traceless part of the second fundamental form of Σ_t. Our theorem shows that conversely every maximal globally hyperbolic spacetimes corresponds to a trajectory of the Moncrief flow. Therefore, maximal globally hyperbolic spacetimes with constant negative curvature and Cauchy surface homeomorphic to Σ are in bijective correspondance with the orbits of the Moncrief flow on $T^*\mathrm{Teich}(\Sigma)$.

Another important interest of Theorem 1.1 is the uniqueness of the CMC time-function τ. In other words, Theorem 1.1 provides a canonical time-function on every maximal globally hyperbolic spacetime with constant negative curvature and compact Cauchy surfaces.

Note that, we already know another canonical time-function on every maximal globally hyperbolic spacetimes with constant negative curvature and compact Cauchy surfaces: the so-called *cosmological time function*. This time function is *regular*, and thus, shares nice properties (it is Lipschitz, admits first and second derivatives almost everywhere, etc., see [2]). Nevertheless, except in very special cases (namely, static spacetimes), the cosmological time function is not differentiable everywhere, whereas the CMC time function provided by Theorem 1.1 is real-analytic.

Benedetti and Bonsante have recently defined a *Wick rotation* using cosmological time functions as a key ingredient. In this context, a Wick rotation is a procedure canonically associating to every spacetime locally modelled on AdS_3 a spacetime locally modelled on Minkowski space Min_3, or a spacetime locally modelled de Sitter space dS_3, or a hyperbolic manifold. One may hope that another Wick rotation (the same ?) could be defined using CMC time functions.

A by-product of the present article is to give new insights into the colossal um-published work of G. Mess. Indeed, a full proof of the classification of globally hyperbolic locally AdS_3 spacetimes, with a new approach and tools, is a important step in the proof of our principal result. It was practically impossible to refer to Mess results without reproducing "everything". Furthermore, we estimated worthwhile and interesting (for the community) to do the point on (at least a part of) Mess work.

Sketch of the proof of Theorem 1.1

Consider a maximal globally hyperbolic (M, g), locally modelled on AdS_3, with compact Cauchy surfaces. The proof of Theorem 1.1 essentially reduces to the existence of a CMC time function τ on M : the uniqueness of this function follows easily from a well-known "maximum principle", and the analiticity of τ follows automatically from the Gauss-Codazzi equation and from the uniqueness of the maximal solution to the Cauchy problem for Einstein equation (see section 2).

In order to prove the existence of τ, we will distinguish two quite different cases according to whether Cauchy surfaces of M have genus 1 (*i.e.* are two-tori), or higher genus (we will see that a Cauchy surface in a locally AdS_3 spacetime cannot be a two-sphere).

In the case where M admits a Cauchy surface of genus 1, we will prove that M is isometric to one of the model spacetimes known has *torus universes* (see [8]). Since such spacetimes are spatially homogeneous, it is quite easy to exhibit explicitely a CMC time function (the level sets of the CMC times function are the orbits of the isometry group of the spacetime). Note that in this case, the CMC time function coïncides with the cosmological time-function. This case is treated in section 7.

The case of spacetimes with higher genus Cauchy surface is more delicate. We first observe that, in this case, the proof of Theorem 1.1 reduces to the existence of a CMC compact spacelike surface in M. Indeed, using Moncrief's flow, and a majoration of the Dirichlet energy of CMC Cauchy surfaces, Andersson, Moncrief and Tromba have proved that the existence of a CMC time function on M follows from the existence of a single CMC Cauchy surface in M (see [4]).

Now, a very classical and general method to prove the existence of CMC surfaces consists in exhibiting a pair of surfaces called "barriers". In our setting, these barriers will be C^2 Cauchy surfaces Σ^-, Σ^+ in M, such that the mean curvature of Σ^+ is everywhere negative, the mean curvature of Σ^- is everywhere positive, and Σ^+ is in the future of Σ^-. It follows e.g. from a result of C. Gerhardt ([11]) that the existence of such barriers implies the existence of a Cauchy surface with constant mean curvature (actually a Cauchy surface with zero mean curvature).

So, we are left to find a pair of barriers in M. The way we construct such barriers is purely geometrical. One of the key ingredients of our proof is the locally projective structure on the anti de Sitter space AdS_3, which provides a notion of convexity. More precisely, using the time orientation and the locally projective structure of AdS_3, we will define some notions of convexity and concavity for spacelike surfaces in M. The key point is that convex (resp. concave) C^2 spacelike surfaces have negative (resp. positive) mean curvature.

Mess' work implies that the spacetime M can be seen as the quotient of a domain U of AdS_3 by a subgroup Γ of $O(2,2)$. We give a very precise description of the domain U. In this description appears naturally a convex set C_0 (roughly speaking, C_0 is the convex hull of the limit set of the group Γ). The boundary of this convex set C_0 is the union of two disjoint Γ-invariant spacelike surfaces which are respectively convex and concave ; the projection Σ_0^- and Σ_0^+ of these surfaces in M are natural candidates to be the barriers.

Unfortunately, the surfaces Σ_0^-, Σ_0^+ are not smooth (only Lipschitz). Smoothness of barriers is an essential requirement in the proof of existence of CMC surfaces. So, the remainder of our proof is devoted to the approximation of the surfaces Σ_0^-, Σ_0^+ by smooth convex and concave spacelike surfaces. Notice that this is not a so easy task as it could appear at first glance : standard convolution methods can not be adapted to our setting (see Remark 6.39).

Remark 1.2. *The notion of* convex hypersurfaces *can be defined in any locally projective space. Hence, the problem raised by the non-smoothness of the surfaces Σ_0^-, Σ_0^+ can be seen as a particular case of a more general question (which, we think, is quite interesting) : Can every (strictly) convex hypersurface in a locally projective space be approximated by a smooth one ?*

2 Uniqueness and analyticity of CMC time functions

The purpose of this section is to prove that, under the hypothesis of Theorem 1.1, the CMC time function τ, if it exists, is unique and real-analytic. First of all, in order to avoid any ambiguity on signs convention, we want to recall the definition of the mean curvature of a spacelike hypersurface in a Lorentzian manifold.

Mean curvature of a spacelike hypersurface.

Let Σ be a smooth spacelike hypersurface in a time-oriented Lorentzian manifold M, and p be a point of Σ. Let n be the future pointing unit normal vector field of S. We recall that the *second fundamental form* of the surface S is the quadratic form II_p on $T_p\Sigma$ defined by $II_p(X,Y) = -g(\nabla_X n, Y)$, where g is the Lorentzian metric and ∇ is the covariant derivative. The *mean curvature* of S at p is the trace of this quadratic form.

Remark 2.1. *Let us identify the tangent space of M at p with $\mathbb{R}^n$, in such a way that the tangent space of Σ at p is identified with $\mathbb{R}^{n-1}\times\{0\}$, and the vector n is identified with $(0,\dots,0,1)$. Let U be a neighbourhood of p in M. If U is small enough, the image of the surface $\Sigma\cap U$ under the inverse of the exponential map $\exp_p$ is the graph of a function $f:\mathbb{R}^{n-1}\to\mathbb{R}$ such that $f(0)=0$ and $Df(0)=0$. The second fundamental form of Σ at p is the opposite of the hessian of f at the origin. In particular, the mean curvature of Σ at p is the opposite of the trace of the hessian of f at the origin.*

Uniqueness of the CMC time function τ

The uniqueness of the time function τ in theorem 1.1 is a particular case of the following result:

Proposition 2.2. *Let M be a globally hyperbolic spacetime with compact Cauchy surfaces. Assume that M admits a CMC time function τ. Then, every compact CMC spacelike surface in M is a fiber of τ.*

Lemma 2.3. *Let Σ and Σ' be smooth spacelike hypersurfaces in a time-oriented Lorentzian manifold M. Assume that Σ and Σ' are tangent at some point p, and assume that Σ' is contained in the future of Σ. Then, the mean curvature of Σ' at p is smaller or equal than those of Σ. Moreover, the mean curvatures of Σ and Σ' at p are equal only if Σ and Σ' have the same 2-jet at p.*

Proof. As in remark 2.1, we identify T_pM with $\mathbb{R}^n$, in such a way that $T_p\Sigma = T_p\Sigma'$ is identified with $\mathbb{R}^{n-1} \times \{0\}$, and the future-pointing unit normal vector of Σ and Σ' at p is identified with $(0, \ldots, 0, 1)$. Let U be a neighbourhood of p in M. If U is small enough, the image of $\Sigma \cap U$ (resp. $\Sigma' \cap U$) under the inverse of the exponential map at p is the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ (resp. of a function $f' : \mathbb{R}^{n-1} \to \mathbb{R}$), such that $f(0) = 0$ and $Df(0) = 0$ (resp. $f'(0) = 0$ and $Df'(0) = 0$). Since Σ' is contained in the future of Σ, we have $f' \geq f$. This implies that, for every $v \in \mathbb{R}^{n-1}$, we have $D^2f'(0).(v, v) \geq D^2f(0)(v, v)$. According to Remark 2.1, this implies that the mean curvature of Σ' at p is smaller or equal than those of Σ'.

The case of equality is a consequence of the following observation: given two functions $f, f' : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f(0) = f'(0) = 0$ and $Df(0) = Df'(0) = 0$, and such that $f' \geq f$, then the hessians of f and f' at p are equal if and only if they have the same trace. □

Proof of the Proposition 2.2. For every $s \in \tau(\mathbb{R})$, denote by Σ_s the Cauchy surface $\tau^{-1}(s)$. Recall that, for every s, Σ_s is a compact Cauchy surface with constant mean curvature equal to s. Now, let $s_1 := \inf\{s \in \mathbb{R} \mid \Sigma \cap \Sigma_s \neq \emptyset\}$ and $s_2 := \inf\{s \in \mathbb{R} \mid \Sigma \cap \Sigma_s \neq \emptyset\}$. The compactness of Σ implies that s_1 and s_2 do exist (*i.e.* are in $\tau(\mathbb{R})$), and that Σ does intersect the surfaces Σ_{s_1} and Σ_{s_2}. Moreover, by definition of s_1 and s_2, the surface Σ is contained in the future the surface Σ_{s_1} and in the past of the surface Σ_{s_2}. Let p_1 be a point in $\Sigma \cap \Sigma_{s_1}$, and p_2 be a point in $\Sigma \cap \Sigma_{s_2}$. By Lemma 2.3, the mean curvature of Σ at p_1 is at most s_1, and the mean curvature of Σ at p_2 is at least s_2. Since Σ is a CMC surface, and since $s_1 \leq s_2$, this implies $s_1 = s_2$. Moreover, since Σ is in the future of Σ_{s_1} and in the past of Σ_{s_2}, this implies $\Sigma = \Sigma_{s_1} = \Sigma_{s_2}$. □

Remark 2.4. *The uniqueness of CMC time function, when it exists, implies that it is preserved by isometries; in particular, by covering automorphisms of isometric coverings. Hence, if a given spacetime admits a CMC time function, the same is true for all its finite quotients. This remark enables us, for the proof of Theorem 1.1, to replace at every moment the spacetime under consideration by any finite covering.*

Analiticity of the CMC time function τ

At first glance, uniqueness of CMC foliations suggests an extra regularity of them. However, uniqueness seems to come from global reasons, and so only an automatic continuity (i.e. C^0 regularity) is guaranteed by general principles. One knows, for instance, many situations in mathematics (e.g. dynamical systems theory) where canonical objects are defined by an infinite limit process, and are therefore never smooth. The situation is better here! The point is that, due to the formalism of the Cauchy problem for Einstein equations, one can have a double vision. The first one is a spacetime endowed with a (local) CMC foliation. The second one is a CMC data, that is, a Riemannian manifold satisfying a "CMC constraint equation", which generates a spacetime having this manifold as a leaf of a CMC foliation. The regularity of the foliation derives thus from that of the associated PDE system. More formally:

Proposition 2.5. *Let (M, g) be an* analytic *Lorentz manifold satisfying vacuum Einstein equation with negative cosmological constant, that is $Ricci_g = \Lambda g$ with $\Lambda < 0$. Let $N \subset M$ be a compact (spacelike) CMC hypersurface. Then, there is a unique CMC foliation extending N, defined on a neighbourhood of it. This foliation is furthermore analytic.*

In particular, any (locally defined) CMC foliation with compact leaves is analytic.

Proof. Firstly, a folkloric fact on Riemannian geometry says that CMC hypersurfaces in analytic manifolds are analytic. The reason is that they solve a quasi-linear elliptic PDE of degree 2. This extends to the Lorentz case.

Now, consider N, a CMC hypersurface in M, and let h and k be its restricted (Riemannian) metric and second fundamental form respectively. Then, (N, h, k) is a CMC vacuum data. See, for instance [3], for a modern exposition on Einstein equations in CMC gauges. The authors write Einstein equation in a gauge which is harmonic on space, and CMC on time. They show that the obtained hyperbolic-elliptic PDE system is well-posed. In particular, solutions are analytic provided that initial data are. □

3 A short presentation of (G, X)-structures

Let X be a manifold and G be a group acting on X with the following property: if an element g of G acts trivially on an open subset of X, then g is the identity element of G. A (G, X)-*structure* on a manifold M is an atlas $(U_i, \varphi_i)_{i\in I}$ where:
– $(U_i)_{i\in I}$ is a covering of M by open subsets,
– for every i, the map φ_i is a homeomorphism from U_i to an open subset of X,
– for every i, j, the transition map $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is the restriction of an element of G.

To every manifold M equipped with a (G, X)-structure are associated two natural objects: the *developping map* $\mathcal{D} : \widetilde{M} \to X$, which is a local homeomorphism from the universal covering $\widetilde{M}$ of M to some open subset of X, and the *holonomy representation* $\rho : \pi_1(M) \to G$. These natural objects satisfy the following equivariance property: for every $x \in \widetilde{M}$ and every $\gamma \in \pi_1(M)$, one has $\mathcal{D}(\gamma.x) = \rho(\gamma).\mathcal{D}(x)$.
A good reference for all these notions is [13].

In this article, we are interested in spacetimes that are *locally modelled* on the anti-de Sitter space AdS_3, that is, manifolds equipped with a (G, X)-structure with $X = AdS_3$ and $G = \mathrm{Isom}_0(AdS_3) = O_0(2, 2)$.

4 The three dimensional anti-de Sitter space

In this section, we recall the construction of the different models of the three-dimensional anti-de Sitter space, and we study the geometrical properties of this space.

4.1 The linear model of the anti-de Sitter space

We denote by (x_1, x_2, x_3, x_4) the standard coordinates on $\mathbb{R}^4$. We will also use the coordinates $(a, b, c, d) = (x_1 - x_3, -x_2 + x_4, x_2 + x_4, x_1 + x_3)$. We consider the quadratic form $Q = -x_1^2 - x_2^2 + x_3^2 + x_4^2 = -ad + bc$ and denote by B_Q the bilinear form associated to Q.

Let p be a point on the quadric of equation $(Q = -1)$ in $\mathbb{R}^4$. When we identify the tangent space of $\mathbb{R}^4$ at p with $\mathbb{R}^4$, the tangent space of the quadric $(Q = -1)$ at p is identified with the Q-orthogonal of p. Since Q is a non-degenerate quadratic form of signature $(-, -, +, +)$, and since $Q(p) = -1$, the restriction of Q to the Q-orthogonal of p is a non-degenerate quadratic form of signature $(-, +, +)$. This proves that the quadratic form Q induces a Lorentzian metric of signature $(-, +, +)$ on the quadric $(Q = -1)$. In other words, the restriction of the pseudo-Riemannian metric $-dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$ to the quadric $(Q = -1)$ is a Lorentzian metric of signature $(-, +, +)$.

Definition 4.1. *The* (linear model of the) three-dimensional anti-de Sitter space*, denoted by* AdS_3*, is the quadric* $(Q = -1)$ *in* $\mathbb{R}^4$ *endowed with the Lorentzian metric induced by* Q*.*

One can easily verify that the anti-de Sitter space AdS_3 is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. More precisely, one can find a diffeomorphism $h : \mathbb{S}^1 \times \mathbb{R}^2 \to AdS_3$ such that the surface $h(\{\theta\} \times \mathbb{R}^2)$ is spacelike for every θ, and such that the circle $h(\mathbb{S}^1 \times \{x\})$ is timelike for every x. In particular, the anti-de Sitter space AdS_3 is time-orientable; from now on, we will assume that a time-orientation has been chosen.

The isometry group of the anti-de Sitter space AdS_3 is the group $O(2,2)$ of the linear transformations of $\mathbb{R}^4$ which preserve the quadratic form Q. The group $O(2,2)$ acts transitively on AdS_3 and the stabilizer of any point is isomorphic to $O(2,1)$; hence, the anti-de Sitter space AdS_3 can be seen as the homogenous space $O(2,2)/O(2,1)$. We shall denote by $O_0(2,2)$ the connected component of the identity of $O(2,2)$; the elements of $O_0(2,2)$ preserve the three-dimensional orientation and the time-orientation of AdS_3.

Proposition 4.2. *The geodesics of AdS_3 are the connected components of the intersections of AdS_3 with the two-dimensional vector subspaces of $\mathbb{R}^4$.*

Proof. Let P be a two-dimensional vector subspace of $\mathbb{R}^4$. The geometry of $P \cap AdS_3$ depends on the signature of the restriction of Q to the plane P:
– If the restriction of Q to the plane P is a quadratic form of signature $(-,-)$, then there exists an element σ of $O(2,2)$ which maps P to the plane $(x_3 = 0, x_4 = 0)$. The intersection of AdS_3 with the plane $(x_3 = 0, x_4 = 0)$ is a closed timelike curve. This curve has to be a geodesic of AdS_3, since it is the fixed point set of the symetry with respect to the plane $(x_3 = 0, x_4 = 0)$, which is an isometry of AdS_3. Hence, the intersection of AdS_3 with the plane P is also a closed timelike geodesic of AdS_3.
– If the restriction of Q to the plane P is a quadratic form of signature $(-,+)$, then there exists an element of $O(2,2)$ which maps P to the plane $(x_1 = 0, x_3 = 0)$. The same arguments as above imply that $P \cap AdS_3$ is the union of two disjoint non-closed spacelike geodesics of AdS_3.
– If the restriction of Q to the plane P is a degenerate quadratic form of signature $(0,-)$, then there exists an element of $O(2,2)$ which maps P to the plane $(x_1 = x_3, x_4 = 0)$. The same arguments as in the first case imply that $P \cap AdS_3$ is a non-closed lightlike geodesic of AdS_3.
– Finally, if the restriction of Q to the plane P is a quadratic form of signature $(+,+)$, $(0,-)$ or $(0,0)$, then one can easily verify that the intersection $P \cap AdS_3$ is empty.

The discussion above implies that each connected component of the intersection of AdS_3 with a 2-dimensional vector subspace of $\mathbb{R}^4$ is a geodesic of AdS_3. The converse follows from the fact that a geodesic is uniquely determined by its tangent vector at some point. $\square$

Remark 4.3. *Let γ be a geodesic of AdS_3. According to Proposition 4.2, there exists a 2-dimensional vector subspace P_γ of $\mathbb{R}^4$ such that γ is a connected component of $P_\gamma \cap AdS_3$. Moreover, reading again the proof of Proposition 4.2, we notice that:*
– if γ is timelike, then the intersection of P_γ with the quadric $(Q = 0)$ is reduced to $(0,0,0,0)$;
– if γ is lightlike, then P_γ is tangent to the quadric $(Q = 0)$ along a line;
– if γ is spacelike, then P_γ intersects transversally the quadric $(Q = 0)$ along two lines.

Remark 4.4. *The proof of Proposition 4.2 shows that all the timelike geodesics of AdS_3 are closed, so that a single point is not an "achronal" set in AdS_3. Moreover, one can prove that the past and the future in AdS_3 of any point $p \in AdS_3$ are both equal to the whole of AdS_3. So, the causal structure of AdS_3 is not very interesting. This is the reason why, instead of working in AdS_3 itself, we shall work in some "large" subsets of AdS_3 which do not contain any closed geodesics (see subsection 4.3).*

Using the same kind of arguments as in the proof of Proposition 4.2, one can prove the following:

Proposition 4.5. *The two-dimensional totally geodesic subspaces of AdS_3 are the connected components of the intersections of AdS_3 with the three-dimensional vector subspaces of $\mathbb{R}^4$.*

Remark 4.6. *In particular, given any point $p \in AdS_3$ and any vector plane P in T_pAdS_3, there exists a totally geodesic subspace of AdS_3 whose tangent space at p is the plane P.*

Let p be a point in AdS_3. We call *dual surface of the point p* the intersection p^* of the hyperplane $p^\perp = \{q \in \mathbb{R}^4 \mid B_Q(p,q) = 0\}$ with AdS_3; hence, by Proposition 4.5, each connected component of p^* is a two-dimensional totally geodesic subspace of AdS_3. One can easily verify that p^* is made of two connected components, and that the restriction of Q to p^* is a quadratic form of signature $(+,+)$ (it is enough to consider the case where p is the point $(1,0,0,0)$ since $O_0(2,2)$ acts transitively on AdS_3). Hence, the surface p^* is the union of two disjoint spacelike totally geodesic subspaces of AdS_3.

Remark 4.7. *Every point of the surface p^* can be joined from p by a timelike geodesic segment.*

Proof. Let q be a point in p^*. We denote by P the 2-dimensional vector subspace spanned by p and q in $\mathbb{R}^4$. We have $Q(p) = Q(q) = -1$ and $B_Q(p,q) = 0$; this implies that the restriction of the quadratic form Q to the plane P is a quadratic form of signature $(-,-)$. Hence, according to the proof of Proposition 4.2, the intersection of the plane P with AdS_3 is a timelike geodesic. This proves in particular that the points p and q are joined by a timelike geodesic segment. $\square$

4.2 The Klein model of the anti-de Sitter space

We shall now define the "Klein model of the anti-de Sitter space". An interesting feature of this model is that it allows us to attach a boundary to the anti-de Sitter space. This boundary will play a fundamental role in the proof of Theorem 1.1.

We see the sphere $\mathbb{S}^3$ as the quotient of $\mathbb{R}^4 \setminus \{0\}$ by positive homotheties. We denote by π the natural projection of $\mathbb{R}^4 \setminus \{0\}$ on $\mathbb{S}^3$. We denote by $[x_1 : x_2 : x_3 : x_4]$ the "positively homogenous" coordinates on $\mathbb{S}^3$ induced by the coordinates (x_1, x_2, x_3, x_4) on $\mathbb{R}^4$: one has $[x_1 : x_2 : x_3 : x_4] = [y_1 : y_2 : y_3 : y_4]$ if and only if there exists $\lambda > 0$ such that $(x_1, x_2, x_3, x_4) = \lambda(y_1, y_2, y_3, y_4)$. Similarly, we denote by $[a : b : c : d]$ the positively homogenous coordinates on $\mathbb{S}^3$ induced by the coordinates (a, b, c, d) on $\mathbb{R}^4$. We endow $\mathbb{S}^3$ with its canonical Riemannian metric.

Remark 4.8. *Given a point $p \in \mathbb{S}^3$, the quantity $Q(p)$ is defined up to multiplication by a positive number; this means that the sign of $Q(p)$ is well-defined. Similarly, given two points $p, q \in \mathbb{S}^3$, the sign of $B_Q(p,q)$ is well-defined.*

Definition 4.9. *The projection π maps diffeomorphically AdS_3 on its image $\pi(AdS_3) \subset \mathbb{S}^3$. The* Klein model of the anti-de Sitter space*, that we denote by $\mathbb{A}d\mathbb{S}_3$, is the image of AdS_3 under π, equipped with the image of the Lorentzian metric of AdS_3. We denote by $\partial\mathbb{A}d\mathbb{S}_3$ the boundary of $\mathbb{A}d\mathbb{S}_3$ in $\mathbb{S}^3$.*

Observe that $\mathbb{A}d\mathbb{S}_3$ is made of the points of $\mathbb{S}^3$ which satisfy the inequation $(Q < 0)$. Hence, $\partial\mathbb{A}d\mathbb{S}_3$ is the quadric of equation $(Q = 0)$ in $\mathbb{S}^3$. This quadric admits two transversal rulings by families of great circles of $\mathbb{S}^3$. The first ruling, that we call *left ruling*, is the family of great circles $\{L_{(\lambda:\mu)}\}_{(\lambda:\mu)\in\mathbb{RP}^1}$ where $L_{(\lambda:\mu)} = \{[a : b : c : d] \in \partial\mathbb{A}d\mathbb{S}_3 \mid (a : c) = (b : d) = (\lambda : \mu) \text{ in } \mathbb{RP}^1\}$. The second ruling, that we call *right ruling*, is the family of great circles $\{R_{(\lambda:\mu)}\}_{(\lambda:\mu)\in\mathbb{RP}^1}$ where $R_{(\lambda:\mu)} = \{[a : b : c : d] \in \partial\mathbb{A}d\mathbb{S}_3 \mid (a : b) = (c : d) = (\lambda : \mu) \text{ in } \mathbb{RP}^1\}$. Through each point of $\partial\mathbb{A}d\mathbb{S}_3$ passes one leaf of the left ruling and one leaf of the right ruling. Any leaf of the left ruling intersects any leaf of the right ruling at two antipodal points.

The elements of $O_0(2,2)$ preserve the left and the right ruling of $\partial\mathbb{A}d\mathbb{S}_3$. Hence, for each element σ of $O_0(2,2)$, we can consider the action of σ on the left and the right rulings. This defines a morphism from $O_0(2,2)$ to $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. It is easy to see that this morphism is onto, and that the kernel of this morphism is a subgroup of order 2 of $O_0(2,2)$. As a consequence, we obtain an isomorphism from $O_0(2,2)$ to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/(-\mathrm{Id},-\mathrm{Id})$ such that the elements of $SL(2,\mathbb{R}) \times \{\pm Id\}/(-\mathrm{Id},-\mathrm{Id})$ preserve individually each circle of the right ruling, and the elements of $\{\pm Id\} \times SL(2,\mathbb{R})/(-\mathrm{Id},-\mathrm{Id})$ preserve individually each leaf of the left ruling.

Proposition 4.10. *The geodesics of $\mathbb{A}d\mathbb{S}_3$ are the connected components of the intersections of $\mathbb{A}d\mathbb{S}_3$ with the great circles of $\mathbb{S}^3$.*

Proof. By construction of $\mathbb{A}d\mathbb{S}_3$, the geodesics of $\mathbb{A}d\mathbb{S}_3$ are the images under π of the geodesics of AdS_3. By Proposition 4.2, the geodesics of AdS_3 are the connected components of the intersections of AdS_3 with the two-dimensional vector subspaces of $\mathbb{R}^4$. The image under π of a two-dimensional vector subspace of $\mathbb{R}^4$ is a great circle of $\mathbb{S}^3$. Putting everything together, we get Proposition 4.10. $\square$

Remark 4.11. *Let γ be a geodesic of $\mathbb{A}d\mathbb{S}_3$. By Proposition 4.10, γ is a connected component of $\mathbb{A}d\mathbb{S}_3 \cap \widehat{\gamma}$, where $\widehat{\gamma}$ is a geodesic of $\mathbb{S}^3$. Moreover, Remark 4.3 and the proof of Proposition 4.10 imply that:*
– if γ is a timelike geodesic, then the great circle $\widehat{\gamma}$ is contained in $\mathbb{A}d\mathbb{S}_3$ and $\gamma = \widehat{\gamma}$,
– if γ is lightlike, then the great circle $\widehat{\gamma}$ is tangent to $\partial\mathbb{A}d\mathbb{S}_3$ at two antipodal points points $p,-p$, and γ is one of the two connected components of $\widehat{\gamma} \setminus \{p,-p\}$,
– if γ is spacelike, then the great circle $\widehat{\gamma}$ intersects $\partial\mathbb{A}d\mathbb{S}_3$ transversally at four points $\{p_1,-p_1,p_2,-p_2\}$, and γ is one of the four connected components of $\widehat{\gamma} \setminus \{p_1,-p_1,p_2,-p_2\}$.

Remark 4.12. *Let q be a point of $\partial\mathbb{A}d\mathbb{S}_3$, and p be a point in $\mathbb{A}d\mathbb{S}_3$. The great 2-sphere S_q of $\mathbb{S}^3$ which is tangent to the quadric $\partial\mathbb{A}d\mathbb{S}_3$ at q is $S_q = \{r \in \mathbb{S}^3 \mid B_Q(q,r) = 0\}$. Consequently, Remark 4.11 implies that there exists a lightlike geodesic γ passing through p and such that the ends of γ in $\partial\mathbb{A}d\mathbb{S}_3$ are the points q and $-q$ if and only if $B_Q(q,p) = 0$.*

Using Proposition 4.5 and the same arguments as in the proof of Proposition 4.10, we obtain:

Proposition 4.13. *The two-dimensional totally geodesic subspaces of $\mathbb{A}d\mathbb{S}_3$ are the connected components of the intersections of $\mathbb{A}d\mathbb{S}_3$ with the great 2-spheres of the sphere $\mathbb{S}^3$.*

Given a point p in $\mathbb{A}d\mathbb{S}_3$, we define the *dual surface $p^\star$ of p* just as we did in the linear model: $p^\star = \{q \in \mathbb{A}d\mathbb{S}_3 \mid B_Q(p,q) = 0\}$. Note that the definitions in the linear model and in the Klein model are coherent: if $\hat{p}$ is a point in AdS_3 such that $\pi(\hat{p}) = p$, then the dual surface of p is the image under π of the dual surface of $\hat{p}$. We denote by $\overline{p^*} = \{q \in \mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3 \mid B_Q(p,q) = 0\}$ the closure on p^* in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$.

Remark 4.14. *In the sequel, we will indifferently denote the anti-de Sitter space by AdS_3 or $\mathbb{A}d\mathbb{S}_3$. Mainly, we will have a preference to the first notation when concerned with metric properties, and to the second one while discussing convexity (see section 4.4) or properties of the boundary at infinity $\partial\mathbb{A}d\mathbb{S}_3$.*

4.3 Affine domains in the anti-de Sitter space

By an *open hemisphere* of $\mathbb{S}^3$, we mean a connected component of $\mathbb{S}^3$ minus a great 2-sphere. Given an open hemisphere U, we say that a diffeomorphism $\varphi : U \to \mathbb{R}^3$ is a *projective chart* if φ maps the great circles of $\mathbb{S}^3$ (intersected with U) to the affine lines of $\mathbb{R}^3$. It is well-known that, for every open hemisphere U of $\mathbb{S}^3$, there exists an projective chart $\varphi : U \to \mathbb{R}^3$. This defines a locally projective structure on $\mathbb{S}^3$, which induces a locally projective structure on $\mathbb{A}d\mathbb{S}_3$. The purpose of this subsection is to define some particular projective charts of $\mathbb{A}d\mathbb{S}_3$.

For every $p \in \mathbb{A}d\mathbb{S}_3$, we consider the open hemisphere $U_p := \{q \in \mathbb{S}^3 \mid B_Q(p,q) < 0\}$, and the sets

$$\begin{array}{rcccl} \mathcal{A}_p & := & \{q \in \mathbb{A}d\mathbb{S}_3 \mid B_Q(p,q) < 0\} & = & \mathbb{A}d\mathbb{S}_3 \cap U_p \\ \partial\mathcal{A}_p & := & \{q \in \partial\mathbb{A}d\mathbb{S}_3 \mid B_Q(p,q) < 0\} & = & \partial\mathbb{A}d\mathbb{S}_3 \cap U_p \end{array}$$

Note that $\partial\mathcal{A}_p$ is *not* the boundary of $\mathcal{A}_p$ in $\mathbb{S}^3$: it is the boundary of $\mathcal{A}_p$ in U_p. Also note that $\mathcal{A}_p$ is the connected component of $\mathbb{A}d\mathbb{S}_3 \setminus p^*$ containing p, and that $\mathcal{A}_p \cup \partial\mathcal{A}_p$ is the connected component of $(\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3) \setminus \overline{p^*}$ containing p.

Let p_0 be the point of coordinates $[1:0:0:0]$ in $\mathbb{S}^3$. We observe that

$$U_{p_0} = \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{S}^3 \mid x_1 > 0\}$$

and we consider the diffeomorphism

$$\begin{array}{rrcl} \Phi_{p_0} : & U_{p_0} & \longrightarrow & \mathbb{R}^3 \\ & [x_1 : x_2 : x_3 : x_4] & \longmapsto & (x,y,z) = \left(\frac{x_3}{x_1}, \frac{x_4}{x_1}, \frac{x_2}{x_1}\right) \end{array}$$

Now, given any point $p \in \mathbb{A}d\mathbb{S}_3$, we can find an element σ_p of $O_0(2,2)$, such that $\sigma_p(p) = p_0$. Then, we consider the diffeomorphism $\Phi_p : U_p \to \mathbb{R}^3$ defined by $\Phi_p = \Phi_{p_0} \circ \sigma_p$.

For every $p \in \mathbb{A}d\mathbb{S}_3$, the diffeomorphism Φ_p maps the domain $\mathcal{A}_p$ on the region of $\mathbb{R}^3$ defined by the inequation $(x^2 + y^2 - z^2 < 1)$, and maps $\partial\mathcal{A}_p$ on the one-sheeted hyperboloid of equation $(x^2 + y^2 - z^2 = -1)$. Moreover, Φ_{p_0} is a projective chart (as the usual stereographic projection), *i.e.* it maps the great circles of $\mathbb{S}^3$ to the affine lines of $\mathbb{R}^3$. Combining this with Proposition 4.10, we obtain that, for every $p \in \mathbb{A}d\mathbb{S}_3$, the diffeomorphism Φ_p maps the geodesics of $\mathbb{A}d\mathbb{S}_3$ to the intersections of the affine lines of $\mathbb{R}^3$ with the set $(x^2 + y^2 - z^2 < 1)$. Similarly, Φ_p maps the totally geodesic subspaces of $\mathbb{A}d\mathbb{S}_3$ to the intersections of the affine planes of $\mathbb{R}^3$ with the set $(x^2 + y^2 - z^2 < 1)$.

Remark 4.15. *Let γ be a geodesic of $\mathbb{A}d\mathbb{S}_3$. Let γ_p be the image under Φ_p of $\gamma \cap \mathcal{A}_p$. According to the above remark, γ_p is contained in an affine line $\widehat{\gamma}_p$ of $\mathbb{R}^3$. Moreover, using Remark 4.11, we see that:*
– if γ is timelike, then the line $\widehat{\gamma}_p$ does not intersect the hyperboloid $(-x^2 + y^2 + z^2 = 1)$ and $\gamma_p = \widehat{\gamma}_p$,
- if γ is lightlike, then the affine line $\widehat{\gamma}_p$ is tangent to the hyperboloid $(-x^2 + y^2 + z^2 = 1)$ at one point q and γ_p is one of the two connected components of $\widehat{\gamma}_p \setminus q$,
- if γ is spacelike, then the line $\widehat{\gamma}_p$ intersects transversally the hyperboloid $(-x^2 + y^2 + z^2 = 1)$ at two points q_1, q_2 and γ is the bounded connected component of $\widehat{\gamma} \setminus \{q_1, q_2\}$.

The image under Φ_p of any geodesic of $\mathbb{A}d\mathbb{S}_3$ is contained in an affine line of $\mathbb{R}^3$. This implies in particular that there is no closed geodesic of $\mathbb{A}d\mathbb{S}_3$ contained in $\mathcal{A}_p$. Moreover, one can prove that there is no closed timelike curve in $\mathcal{A}_p$, so that the causal structure of $\mathcal{A}_p$ is more interesting than those of $\mathbb{A}d\mathbb{S}_3$ (see Remark 4.4).

4.4 Convex subsets of $\mathbb{A}d\mathbb{S}_3$

Using the local projective structure of $\mathbb{A}d\mathbb{S}_3$, we will define a notion of convex subsets of $\mathbb{A}d\mathbb{S}_3$.

First, we define a *convex subset of* $\mathbb{S}^3$ to be a set $C \subset \mathbb{S}^3$ such that: C is contained in some open hemisphere U of $\mathbb{S}^3$, and there exists some projective chart $\varphi : U \to \mathbb{R}^3$ such that the set $\varphi(C)$ is a convex subset of $\mathbb{R}^3$.

Note that, if C is a convex subset of $\mathbb{S}^3$, then, for *every* open hemisphere V of $\mathbb{S}^3$ containing C, and *every* projective chart $\psi : V \to \mathbb{R}^3$, the set $\varphi(C)$ is a convex subset of $\mathbb{R}^3$. Moreover, a set C contained in some open hemisphere of $\mathbb{S}^3$ is a convex subset of $\mathbb{S}^3$ if and only if the positive cone $\pi^{-1}(C)$ is a convex subset of $\mathbb{R}^4$ (recall that π is the natural projection of $\mathbb{R}^4 \setminus \{0\}$ on $\mathbb{S}^3$).

Now, given a subset E of $\mathbb{S}^3$ such that C is contained in some open hemisphere of $\mathbb{S}^3$, we define the *convex hull* $\mathrm{Conv}(C)$ of the set C to be the intersection of all the convex subsets of $\mathbb{S}^3$ containing C. Note that, if U is an open hemisphere containing C and $\Phi : U \to \mathbb{R}^3$ is a projective chart, the set $\mathrm{Conv}(C)$ is the image under Φ^{-1} of the convex hull in $\mathbb{R}^3$ of the set $\Phi(C)$. Moreover, $\mathrm{Conv}(C)$ is also the image under π of the convex hull in $\mathbb{R}^4$ of the positive cone $\pi^{-1}(C)$.

Now, recall that $\mathbb{A}d\mathbb{S}_3$ is contained in the sphere $\mathbb{S}^3$, and let C be a subset of $\mathbb{A}d\mathbb{S}_3$. We say that C is a *convex subset of* $\mathbb{A}d\mathbb{S}_3$ if it is convex as a subset of $\mathbb{S}^3$. We say that C is a *relatively convex subset C of* $\mathbb{A}d\mathbb{S}_3$ if C is the intersection of $\mathbb{A}d\mathbb{S}_3$ with a convex subset of $\mathbb{S}^3$. Equivalently, C is a convex subset of $\mathbb{A}d\mathbb{S}_3$ if $C = \mathrm{Conv}(C)$, and C is a relatively convex subset of $\mathbb{A}d\mathbb{S}_3$ if $C = \mathrm{Conv}(C) \cap \mathbb{A}d\mathbb{S}_3$.

4.5 The $SL(2,\mathbb{R})$-model of the anti-de Sitter space

The linear model of the 3-dimensional anti-de Sitter space is the quadric $\{(a,b,c,d) \in \mathbb{R}^4 \mid -ad+bc = -1\}$ endowed with the Lorentzian metric induced by the quadratic form $Q(a,b,c,d) = -ad+bc$. Therefore, the anti-de Sitter space can be identified with the group of matrices $SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) \mid ad - bc = 1 \right\}$ endowed with the Lorentzian metric induced by the quadratic form $-\det$ defined on $M(2,\mathbb{R})$ by $-\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

The quadratic form $-\det$ on $M(2,\mathbb{R})$ is invariant under left and right multiplication by elements of $SL(2,\mathbb{R})$ (actually, the Lorentzian metric induced by $-\det$ is a multiple of the Killing form of the Lie group $SL(2,\mathbb{R})$). This implies that the isometry group of $(SL(2,\mathbb{R}), -\det)$ is $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ acting on $SL(2,\mathbb{R})$ by left and right multiplication, *i.e.* acting by $(g_1,g_2).g = g_1 g g_2^{-1}$.

4.6 Causal structure of the anti-de Sitter space

Denote dt^2 the standard Riemannian metric on the circle $\mathbb{S}^1$, by ds^2 the standard Riemannian metric on the 2-dimensional sphere $\mathbb{S}^2$, by $\mathbb{D}^2$ the open upper-hemisphere of $\mathbb{S}^2$, and by $\overline{\mathbb{D}^2}$ the closure of $\mathbb{D}^2$. We will prove that $\mathbb{A}d\mathbb{S}_3$ has the same causal structure as $(\mathbb{S}^1 \times \mathbb{D}^2, -dt^2 + ds^2)$. More precisely:

Proposition 4.16. *There exists a diffeomorphism $\Psi : \mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1 \times \mathbb{D}^2$ such that the pull back by Ψ of the Lorentzian metric $-dt^2+ds^2$ defines the same causal structure as the original metric of $\mathbb{A}d\mathbb{S}_3$, that is, the two metrics are in the same conformal class. Moreover, the diffeomorphism Ψ can be extended to a diffeomorphism $\overline{\Psi} : \mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1 \times \overline{\mathbb{D}^2}$.*

To prove this, we will embed $\mathbb{A}d\mathbb{S}_3$ in the so-called *three-dimensional Einstein universe.* Denote by (x_1,x_2,x_3,x_4,x_5) the standard coordinates on $\mathbb{R}^5$, consider the quadratic form $\widetilde{Q}$ on $\mathbb{R}^5$ defined by $\widetilde{Q}(x_1,x_2,x_3,x_4,x_5) = -x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2$, denote by $\mathbb{S}^4$ the quotient of $\mathbb{R}^5 \setminus \{0\}$ by positive homotheties, and by $\widetilde{\pi}$ the natural projection of $\mathbb{R}^5 \setminus \{0\}$ on $\mathbb{S}^4$. Then, the *three-dimensional Einstein space*, denoted by Ein_3, is the image under $\widetilde{\pi}$ of the quadric $(\widetilde{Q} = 0)$. There is a natural conformal class of Lorentzian metrics on Ein_3, defined as follows:

— Given an open subset U of Ein_3, and a local section $\sigma : U \to \mathbb{R}^5 \setminus \{0\}$ of the projection $\widetilde{\pi}$, we define a Lorentzian metric g_σ on U as follows. For every point $p \in U$ and every vector $v \in T_p Ein_3$, we choose a vector $\hat{v} \in T_{\sigma(p)}\mathbb{R}^5$ such that $d\widetilde{\pi}(\sigma(p)).\hat{v} = v$. The quantity $\widetilde{Q}(\hat{v})$ does not depend on the choice of the vector $\hat{v}$: indeed, the vector $\hat{v}$ is tangent to the quadric $(\widetilde{Q} = 0)$, the vector $\hat{v}$ is defined up to the addition of an element of $\widetilde{\pi}^{-1}(p)$, and the half-line $\widetilde{\pi}^{-1}(p)$ is contained in the $\widetilde{Q}$-orthogonal of the tangent space of the quadric $(\widetilde{Q} = 0)$ at σ_p. We set $g_\sigma(v) := \widetilde{Q}(\hat{v})$.

— The conformal class of the metric g_σ does not depend on the section σ. Indeed, if σ and σ' are two sections of the projection $\widetilde{\pi}$ defined on U, then we have $g_{\sigma'} = \lambda^2.g_\sigma$, where $\lambda : U \to \mathbb{R}$ is the function such that $\sigma' = \lambda.\sigma$.

Proof of Proposition 4.16. Let $A = \{[x_1 : x_2 : x_3 : x_4 : x_5] \in Ein_3 \mid x_5 > 0\}$, and let ∂A be the boundary of A. We will consider two particular sections of the projection $\widetilde{\pi}$. First, we consider the section σ, defined on A, whose image is contained in the affine hyperplane $x_5 = 1$. The anti-de Sitter space $\mathbb{A}d\mathbb{S}_3$ is isometric to the set A equipped with the Lorentzian metric g_σ: the most natural isometry is the diffeomorphism Φ defined by $\Phi([x_1 : x_2 : x_3 : x_4]) = [x_1 : x_2 : x_3 : x_4 : 1]$. Now, we consider the section σ', defined on the whole of Ein_3, whose image is contained in the Euclidean sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 2$. The set A equipped with the Lorentzian metric $g_{\sigma'}$ is isometric to the set $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 = 1 \ , \ x_3^2 + x_4^2 + x_5^2 = 1 \ , \ x_5 > 0\} \simeq \mathbb{S}^1 \times \mathbb{D}^2$ equipped with the Lorentzian metric $-(dx_1^2 + dx_2^2) + (dx_3^2 + dx_4^2 + dx_5^2) \simeq -dt^2 + ds^2$: the most natural isometry is the diffeomorphism $\Phi' = \sigma'_{|A}$. We consider the diffeomorphism $\Psi := \Phi' \circ \Phi : \mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1 \times \mathbb{D}^2$. Since the metric g_σ and $g_{\sigma'}$ are conformally equivalent, the pull back by Ψ of the metric $-dt^2 + ds^2$ is conformally equivalent to the original metric of $\mathbb{A}d\mathbb{S}_3$.

The diffeomorphism Φ can be extended to a diffeomorphism $\overline{\Phi} : \mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3 \to A \cup \partial A$: for every $[x_1 : x_2 : x_3 : x_4]$ in $\partial\mathbb{A}d\mathbb{S}_3$, we have $\overline{\Phi}([x_1 : x_2 : x_3 : x_4]) = [x_1 : x_2 : x_3 : x_4 : 0]$. The diffeomorphism Φ' can be extended to a diffeomorphism $\overline{\Phi'} : A \cup \partial A \to \mathbb{S}^1 \times \overline{\mathbb{D}^2}$: we have $\overline{\Phi'} = \sigma'_{A \cup \partial A}$. Hence, the diffeomorphism Ψ can be extended to a diffeomorphism $\overline{\Psi} = \overline{\Phi} \circ \overline{\Phi'} : \mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1 \times \overline{\mathbb{D}^2}$. □

Causal structure on $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. Let $\overline{g}$ be the Lorentzian metric on $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$, obtained by pulling back the Lorentzian metric $-dt^2 + ds^2$ defined on $\mathbb{S}^1 \times \overline{\mathbb{D}^2}$ by the diffeomorphism $\overline{\Psi}$. The Lorentzian metric $\overline{g}$ defines the same causal structure on $\mathbb{A}d\mathbb{S}_3$ as the original metric of $\mathbb{A}d\mathbb{S}_3$. From now on, we endow $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ with the causal structure defined by the metric $\overline{g}$. This causal structure allows us to speak of timelike, lightlike and spacelike objects in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. In particular, we can consider the causal structure induced on the quadric $\partial\mathbb{A}d\mathbb{S}_3$. Given a point $q \in \partial\mathbb{A}d\mathbb{S}_3$, it is easy to verify that the lightcone of q for this conformally Lorentzian structure is the union of the leaf of the left ruling and of the circle of the right ruling passing through q.

Remark 4.17. *Let p_0 be the point of coordinates $[1 : 0 : 0 : 0]$ in $\mathbb{S}^3$. Recall that $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$ is the subset of $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ defined by the inequation $(x_1 > 0)$. Hence, the diffeomorphism $\overline{\Psi}$ defined above maps $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$ on $\{(x_1, x_2, x_3, x_4, x_5) \mid x_1^2 + x_2^2 = 1 \ , \ x_1 > 0 \ , \ x_3^2 + x_4^2 + x_5^2 = 1 \ , \ x_5 \geq 0\} \simeq (-\pi/2, \pi/2) \times \overline{\mathbb{D}^2}$.*

Corollary 4.18. *For every $p \in \mathbb{A}d\mathbb{S}_3$, the domain $\mathcal{A}_p \cup \partial\mathcal{A}_p$ has the same causal structure as the Lorentzian space $\Big((-\pi/2, \pi/2) \times \overline{\mathbb{D}^2}, -dt^2 + ds^2\Big)$.*

Proof. Since $O(2,2)$ acts transitively on $\mathbb{A}d\mathbb{S}_3$, it is enough to consider the case where p is the point of coordinates $[1 : 0 : 0 : 0]$. This case follows from Proposition 4.16 and Remark 4.17. □

The two following propositions will play some fundamental roles in the proof of Theorem 1.1:

Proposition 4.19. *Let p be a point in $\mathbb{A}d\mathbb{S}_3$, and q be a point in $\partial\mathcal{A}_p$. A point $r \in \mathcal{A}_p \cup \partial\mathcal{A}_p$ can be joined from q by a timelike (resp. causal) curve if and only if $B_Q(q, r)$ is positive (resp. non-negative).*

Proof. Since $O(2,2)$ acts transitively on $\mathbb{A}d\mathbb{S}_3$, we can assume that $p = [1 : 0 : 0 : 0]$. There exists a timelike curve joining q to r in $\mathcal{A}_p \cup \partial\mathcal{A}_p$ if and only if there exists a timelike curve joining $\Psi_p(q)$ to $\Psi_p(r)$ in $\Big((-\pi/2, \pi/2) \times \overline{\mathbb{D}^2}, -dt^2 + ds^2\Big)$. We see $\Big((-\pi/2, \pi/2) \times \overline{\mathbb{D}^2}, -dt^2 + ds^2\Big)$ as the set $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1^2+x_2^2 = 1 \ , \ x_1 > 0 \ , \ x_3^2+x_4^2+x_5^2 = 1 \ , \ x_5 > 0\}$ equipped with the metric $-(dx_1^2+dx_2^2)+(dx_3^2+dx_4^2+dx_5^2)$. Coming back to the definition of the diffeomorphism Ψ_p (see the proof of Proposition 4.16), we observe that $B_Q(q, r)$ and $B_{\widetilde{Q}}(\Psi_p(q), \Psi_p(r))$ have the same sign. Moreover, it is clear that the points $\Psi_p(q)$ and $\Psi_p(r)$ can be joined by a timelike (resp. causal) curve in $\left((-\pi/2, \pi/2) \times \overline{\mathbb{D}^2}, -dt^2 + ds^2\right)$ if and only if $\widetilde{Q}(\Psi_p(q)-\Psi_p(r))$ is negative (resp. non-positive). Finally, notice that the quantity $\widetilde{Q}(\Psi_p(q) - \Psi_p(r))$ and $B_{\widetilde{Q}}(\Psi_p(q), \Psi_p(r))$ have opposite signs (since $\widetilde{Q}(\Psi_p(q)) = \widetilde{Q}(\Psi_p(r)) = 0$). Putting everything together, we obtain the proposition. □

Remark 4.20. *Let p be a point in $\mathbb{A}d\mathbb{S}_3$. Let P be a totally geodesic spacelike subspace of $\mathcal{A}_p$ (by such we mean the intersection of $\mathcal{A}_p$ with a totally geodesic spacelike subspace of $\mathbb{A}d\mathbb{S}_3$). Then, P divides $\mathcal{A}_p$ into two closed regions: the past of P in $\mathcal{A}_p$ and the future of P in $\mathcal{A}_p$.*

Proof. We identify $\mathcal{A}_p$ and P with their images under the embedding Φ_p. Then, P is the intersection of $\mathcal{A}_p$ (*i.e.* of the set $(-x^2 + y^2 + z^2 < 1)$) with an affine plane $\widehat{P}$ of $\mathbb{R}^3$. We consider the two regions of $\mathcal{A}_p$ defined as the intersections of $\mathcal{A}_p$ with the closures two connected components of $\mathbb{R}^3 \setminus \widehat{P}$. Since P is spacelike and connected, the past (resp. the future) of P in $\mathcal{A}_p$ is necessarily contained in one of these two regions. Finally, Remark 4.15 implies that, for every point $q \in \mathcal{A}_p$, there exists a timelike geodesic joining q to a point of P. Hence, the union of the past and the future of P must be equal to $\mathcal{A}_p$. The proposition follows. □

5 Globally hyperbolic spacetimes

All along this section, we consider a maximal globally hyperbolic spacetime M, locally modelled on AdS_3, with closed orientable Cauchy surfaces. All the Cauchy surfaces have the same genus, that we denote by g. We denote by $\widetilde{M}$ the universal covering of M. We choose a Cauchy surface Σ_0 in M, and the lift $\widetilde{\Sigma}_0$ of Σ_0 in $\widetilde{M}$. Since M is locally modelled on AdS_3, we can consider the developping map $\mathcal{D} : \widetilde{M} \to AdS_3$ and the holonomy representation $\rho : \pi_1(M) = \pi_1(\Sigma_0) \to O_0(2, 2)$ (see section 3).

Let $S_0 = \mathcal{D}(\widetilde{\Sigma}_0)$, and $\Gamma = \rho(\pi_1(M))$. Identifying $O_0(2, 2)$ with $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})/(-\mathrm{Id}, -\mathrm{Id})$ (see subsection 4.2), we can see ρ as a representation of $\pi_1(M)$ in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Then, we will denote by ρ_L and ρ_R the representations of $\pi_1(M)$ in $SL(2, \mathbb{R})$ such that $\rho = \rho_L \times \rho_R$.

In subsection 5.1, we will study the surface S_0 and its boundary ∂S_0 in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. In particular, we will show that S cannot be a sphere, i.e., its genus g is positive. The results of this subsection are not original: most of them are contained in Mess preprint ([15]). Yet, we will provide a proof of each result to keep our paper as self-contained as possible (by the way, using the conformal equivalence of $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ with $(\overline{\mathbb{D}}^2 \times \mathbb{S}^1, -dt^2 + ds^2)$, we were able to simplify some of the proofs of Mess).

In subsection 5.2, we study the Cauchy development $D(S_0)$ of the surface S_0. In particular, we prove that M is isometric to the quotient $\Gamma \backslash D(S_0)$.

5.1 The spacelike surface S_0

The purpose of this subsection is to collect as many information as possible on the surface S_0. In particular, we will prove that S_0 is an open disc properly embedded in $\mathbb{A}d\mathbb{S}_3$, that the closure $\overline{S_0}$ of S_0 in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ is a closed topological disc, and that $\overline{S_0}$ is an achronal set.

The Lorentzian metric of M induces a Riemannian metric on the Cauchy surface Σ_0, which can be lifted to get a Riemannian metric on $\widetilde{\Sigma}_0$. Since Σ_0 is compact, the Riemannian metrics on Σ_0 and $\widetilde{\Sigma}_0$ are complete. The developping map $\mathcal{D}$ induces a locally isometric immersion of the surface $\widetilde{\Sigma}_0$ in AdS_3. It turns out that this immersion is automatically a proper embedding:

Proposition 5.1. *The surface S_0 is an open disc properly embedded in AdS_3. Moreover, every timelike geodesic of AdS_3 intersects the surface S_0 at exactly one point.*

Proof. We consider the projection $\zeta : AdS_3 \to \mathbb{R}^2$, defined by $\zeta(x_1, x_2, x_3, x_4) = (x_3, x_4)$. Observe that the fibers of the projection ζ are the orbits of a timelike killing vector field of $\mathbb{A}d\mathbb{S}_3$. We endow $\mathbb{R}^2$ with the Riemannian metric g_ζ defined as follows. Given a point $q \in \mathbb{R}^2$ and a vector $v \in T_q\mathbb{R}^2$, we choose a point $\hat{q} \in \zeta^{-1}(q)$, and we consider the unique vector $\hat{v} \in T_{\hat{q}}AdS_3$ such that $d\zeta_{\hat{q}}.\hat{v} = v$ and such that $\hat{v}$ is orthogonal to the fibers $\zeta^{-1}(q)$. We define $g_\zeta(v)$ to be to the norm of the vector $\hat{v}$ for the Lorentzian metric of AdS_3. This definition does not depend on the choice of the point $\hat{q}$, since the fibers of ζ are the orbits of a killing vector field. It is easy to verify that $\mathbb{R}^2$ endowed with the metric g_ζ is isometric to the hyperbolic plane.

Claim 1. Given any point $q \in AdS_3$ and any spacelike vector v in T_qAdS_3, the norm of the vector $d\zeta_q(v)$ for the metric g_ζ is bigger than the norm of v in AdS_3.

Indeed, write $v = u + w$ where u is tangent to the fiber of the projection ζ (in particular, u is timelike) and w is orthogonal to this fiber. On the one hand, by definition of g_ζ, the norm of the vector $d\zeta_q(v)$ for the metric g_ζ is equal to the norm of w in AdS_3. On the other hand, the norm of v in AdS_3 is less than the norm of w, since u is timelike. This completes the proof of claim 1.

Claim 2. For every locally isometric immersion $f : \widetilde{\Sigma}_0 \to AdS_3$, the map $\zeta \circ f : \widetilde{\Sigma}_0 \to \mathbb{R}^2$ is an homeomorphism. In particular, the surface $f(\widetilde{\Sigma}_0)$ intersects each fiber of ζ at exactly one point.

By the first claim, the map $\zeta \circ f$ is locally distance increasing (when the surface $\widetilde{\Sigma}_0$ is endowed with its Riemannian metric, and $\mathbb{R}^2$ is endowed with the metric g_ζ). Since the Riemannian metric of Σ_0 is complete, this implies that $\zeta \circ f : \widetilde{\Sigma}_0 \to \mathbb{R}^2$ has the path lifting property, and thus is a covering map. Since H is simply connected, this implies that $\zeta \circ f : \widetilde{\Sigma}_0 \to \mathbb{R}^2$ is an homeomorphism. This completes the proof of claim 2.

Applying claim 2 with f being the developping map $\mathcal{D}$, we obtain that $\mathcal{D} : \widetilde{\Sigma}_0 \to AdS_3$ is a proper embedding, and that $\widetilde{\Sigma}_0$ is homeomorphic to $\mathbb{R}^2$ (and thus homeomorphic to an open disc). Hence, the surface $S_0 := \mathcal{D}(\widetilde{\Sigma}_0)$ is an open disc properly embedded in AdS_3. Now, let γ be a timelike geodesic of AdS_3. Observe that the circle $\zeta^{-1}(0,0)$ is a timelike geodesic of AdS_3. Since $O(2,2)$ acts transitively on the set of timelike geodesic of AdS_3, there exists $\sigma \in O(2,2)$ such that $\sigma(\gamma) = \zeta^{-1}(0,0)$; in particular, $\sigma(\gamma)$ is a fiber of the projection ζ. Applying claim 2 with $f = \sigma^{-1} \circ \mathcal{D}$, we obtain that the surface $\sigma^{-1}(S_0) = \sigma^{-1} \circ \mathcal{D}(\widetilde{\Sigma}_0)$ intersects each fiber of ζ at exactly one point. Hence, the surface S_0 intersects the geodesic γ at exactly point. $\square$

Remark 5.2. *Proposition 5.1 is still valid if Σ_0 is replaced by another Cauchy surface of M.*

Remark 5.3. *The proof of Proposition 5.1 shows that $\widetilde{\Sigma}_0$ is homeomorphic to a disc. Hence, there does not exist any globally hyperbolic spacetime, locally modelled on AdS_3, with closed orientable Cauchy surfaces of genus 0.*

Now, we will use the conformal equivalence between $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ and $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$. Let us start by some remarks:

Remark 5.4. *(i) Let S be a spacelike (resp. non-timelike) surface in $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$. Then, every point of S has a neighbourhood in S which is the graph of a contracting*[1] *(resp. 1-Lipschitz) mapping $f : (U, ds^2) \to (\mathbb{S}^1, dt^2)$, where U is an open subset of $\overline{\mathbb{D}^2}$.*

[1]We recall that, given two metric spaces (E, d) and (E', d'), a mapping $f : (E, d) \to (E', d')$ is said to be *contracting* if $d'(f(x), f(y)) < d(x, y)$ for every $x \neq y$.

(ii) Every properly embedded spacelike (resp. non timelike) surface in $(\mathbb{S}^1 \times \mathbb{D}^2, -dt^2 + ds^2)$ is the graph of a contracting (resp. 1-Lipschitz) mapping $f : (\mathbb{D}^2, ds^2) \to (\mathbb{S}^1, dt^2)$.

(iii) Of course, (i) and (ii) remain true if we replace $\mathbb{S}^1$ by $(-\pi/2, \pi/2)$.

Proof. Item (i) is an immediate consequence of the product structure of $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$. To prove (ii), we consider a properly embedded spacelike (resp. non-timelike) surface S in $(\mathbb{S}^1 \times \mathbb{D}^2, -dt^2 + ds^2)$. Let p_2 be the projection of $\mathbb{S}^1 \times \mathbb{D}^2$ on $\mathbb{D}^2$. Using item (i) and the fact that S is properly embedded, it is easy to show that $p_2 : S \to \mathbb{D}^2$ is a covering map. Hence, $p_2 : S \to \mathbb{D}^2$ is a homeomorphism, and the surface S is the graph of a mapping $f : \mathbb{D}^2 \to \mathbb{S}^1$. By item (i), the mapping f is contracting (resp. 1-Lipschitz). □

Remark 5.5. *In the same vein, we observe that timelike (resp. causal) curves are represented in $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$ by graphs of contracting (resp. 1-Lipschitz) mappings $g : (J, dt^2) \to (\overline{\mathbb{D}^2}, ds^2)$, where J is a subinterval of $\mathbb{S}^1$.*

Putting Proposition 5.1 and Remark 5.4 together, we obtain the following:

Proposition 5.6. *Any conformal equivalence between $\mathbb{A}d\mathbb{S}_3$ and $(\mathbb{S}^1 \times \mathbb{D}^2, -dt^2 + ds^2)$ maps the surface S_0 to the graph of a contracting mapping $f : \mathbb{D}^2 \to \mathbb{S}^1$.*

Now, let us denote by $\overline{S_0}$ the closure of the surface S_0 in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$.

Corollary 5.7. *Any conformal equivalence between $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ and $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$ maps the closure $\overline{S_0}$ of the surface S_0 to the graph of a 1-Lipschitz mapping $\overline{f} : (\overline{\mathbb{D}^2}, ds^2) \to (\mathbb{S}^1, dt^2)$, which is contracting in restriction to the open disc $\mathbb{D}^2$. In particular, $\overline{S}_0$ is a closed topological disc.*

Proof. The result follows from Proposition 5.6 and from the fact that any contracting mapping from $(\mathbb{D}^2, ds^2)$ to $(\mathbb{S}^1, dt^2)$ can be extended as a 1-Lipschtiz mapping from $(\overline{\mathbb{D}^2}, ds^2)$ to $(\mathbb{S}^1, dt^2)$. □

Proposition 5.1 and corollary 5.7 imply that the boundary ∂S_0 of the surface S_0 in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ is a topological simple closed curve contained in $\partial\mathbb{A}d\mathbb{S}_3$. Of course, the curve ∂S_0 must be invariant by the holonomy group $\Gamma = \rho(\pi_1(M))$.

Remark 5.8. *According to the proof of Proposition 5.1, the surface S_0 intersects each fiber of the projection $\zeta : AdS_3 \to \mathbb{R}^2$ defined by $\zeta((x_1, x_2, x_3, x_4)) = (x_3, x_4)$. This implies that the curve ∂S_0 intersects each fiber of the projection $\zeta : \partial\mathbb{A}d\mathbb{S}_3 \to \mathbb{S}^1$ defined by $\zeta([x_1 : x_2 : x_3 : x_4]) = [x_3 : x_4]$.*

Futhermore, if we identify $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ with $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$, then the curve ∂S_0 is identified with the graph of a mapping from $\partial\mathbb{D}^2$ to $\mathbb{S}^1$. This implies, in particular, that the curve ∂S_0 is not null-homotopic in $\partial\mathbb{A}d\mathbb{S}_3$.

Thanks to Remark 5.4, we can define a notion of spacelike topological surface in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$:

Definition 5.9. *Let S be a topological surface (with or without boundary) in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. Using the conformal equivalence between $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ and $(\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$, we can see S as a surface in $\mathbb{S}^1 \times \overline{\mathbb{D}^2}$. We will say that the topological surface S is* spacelike *(resp.* non timelike*) if every point of S has a neighbourhood in S which is the graph of a contracting (resp. 1-Lipschitz) mapping $f : (U, ds^2) \to (\mathbb{S}^1, dt^2)$, where U is an open subset of $\overline{\mathbb{D}^2}$.*

With this definition, $\overline{S_0}$ is a non-timelike topological surface in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$.

Proposition 5.10. *Every lightlike geodesic intersects the surface S_0 at most once. Moreover, if a lightlike geodesic has one of its endpoints on the curve ∂S_0, then this geodesic does not intersect S_0.*

Proof. Let p be a point on the surface S_0, and γ a lightlike geodesic containing p. Denote by d the distance function on the hemisphere $\overline{\mathbb{D}}^2$, and let p_0 be the center of the hemisphere, i.e. the unique point for which $d(p_0, q) = \pi/2$ for any point q in $\partial\mathbb{D}^2$. Select a conformal equivalence $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3 \approx (\mathbb{S}^1 \times \overline{\mathbb{D}^2}, -dt^2 + ds^2)$ for which p is identified with $(0, p_0)$ and $\mathcal{A}_p$ with $]-\pi/2, \pi/2[\times\mathbb{D}^2$. Then, $\overline{S_0}$ is represented as the graph of a 1-Lipschitz mapping f for which $f(p_0) = 0$. On the other hand, as every lightlike geodesic containing p, γ is contained in $\mathcal{A}_p$ and is represented by a curve $(d(p_0, r), r)$, where r describes a geodesic in $\mathbb{D}^2$ containing p_0. Since the restriction of f to $\mathbb{D}^2$ is contracting, it follows immediatly that γ does not contain another point of S_0 than p. The first statement in the proposition follows.

Assume now that one of the two end points of γ is $(f(q), q) \in \partial S_0$. Then, $d(q, p_0) = \pi/2 = f(q)$, and since f is 1-Lipschitz, for any point r on the geodesic of $\mathbb{D}^2$ under consideration, we must have $d(p_0, r) = f(r)$. This is impossible, since the restriction of f to $\mathbb{D}^2$ is contracting. □

Proposition 5.11. *For every $p \in S_0$, the surface $\overline{S_0}$ is contained in the affine domain $\mathcal{A}_p \cup \partial\mathcal{A}_p$.*

Proof. We keep the notation used in the proof of the previous lemma. It follows immediatly that the maximum value of f is at most $\pi/2$, and its minimum value is at least $-\pi/2$. In other words, $\overline{S}_0$ is contained in the closure of $\mathcal{A}_p$. Moreover, in the proof above we have actually shown that f does not attain the values $\pi/2$, $-\pi/2$. The proposition follows. □

Proposition 5.12. *For every $p \in \mathbb{A}d\mathbb{S}_3$ such that $\overline{S_0} \subset \mathcal{A}_p \cup \partial\mathcal{A}_p$, the surface $\overline{S_0}$ is an achronal subset of $\mathcal{A}_p \cup \partial\mathcal{A}_p$ (*i.e. *a timelike curve contained in $\mathcal{A}_p \cup \partial\mathcal{A}_p$ cannot intersect $\overline{S_0}$ at two distinct points). Moreover, if two points in $\overline{S_0}$ are causally related, then they belong to a lightlike geodesic of $\partial\mathbb{A}d\mathbb{S}_3$ contained in $\partial\overline{S_0}$.*

Proof. We keep the notations used in the proof of Proposition 5.10 (except that $(0, p_0)$ is not assumed now to belong to S_0, i.e., the mapping f admitting $\overline{S_0}$ as graph does not necessarly vanish at p_0). A future oriented causal curve in $\mathcal{A}_p$ is represented by a curve $(g(t), r(t))$ where g satisfies: $g(t) - g(s) \geq d(r(t), r(s))$. Assume the existence of $t < t'$ such that $g(t) = f(r(t))$ and $g(t') = f(r(t'))$. Then:

$$\mid f(r(t')) - f(r(t)) \mid \leq d(r(t), r(t')) \leq g(t') - g(t) = f(r(t')) - f(r(t))$$

Therefore, all these inequations are equalities. According to Proposition 5.10, it follows that $(g(t), r(t))$ and $(g(t'), r(t'))$ belong both to $\mathcal{A}_p$. Moreover, it follows that for every s in $[t, t']$, $f(r(s)) = g(r(s)) = f(r(t)) + d(r(s), r(t))$. The proposition follows. □

Remark 5.13. *Let p be a point such that the surface S_0 is contained in $\mathcal{A}_p$. Proposition 5.1 implies that every point of $\mathcal{A}_p$ is either in the past*[2] *or in the future of the surface S_0. Moreover, it should be clear to the reader that, according to corollary 5.7 and Proposition 5.12, a point of $\mathcal{A}_p$ cannot be simultaneously in the past and in the future of the surface, except if it is on the surface S_0.*

5.2 Cauchy development of the surface S_0

In this subsection, we study the Cauchy development $D(S_0)$ of the surface S_0 in AdS_3. The main goal of the subsection is to prove that M is isometric to a quotient $\Gamma\backslash D(S_0)$.

Let us first recall the definition of the Cauchy development of a spacelike surface. Given a spacelike surface S in AdS_3, the *past Cauchy development* $D^-(S)$ of S is the set of all points $p \in AdS_3$ such that every future-inextendable causal curve through p intersects S. The *future*

[2]Here, by "past", we mean the "past in $\mathcal{A}_p$": a point q is in the past of the surface S_0 if there exists a future-directed causal curve contained in $\mathcal{A}_p$ going from S_0 to q. Similarly for the future.

Cauchy development $D^+(S)$ of S is defined similarly. The *Cauchy development* of S is the set $D(S) := D^-(S) \cup D^+(S)$. It is well-known and not difficult to prove that $D(S)$ is a connected open domain. The following lemma provides a more tractable definition of $D(S)$:

Lemma 5.14. *Let $S \subset AdS_3$ be a spacelike surface. The past Cauchy development of S is the set of all points p such that every inextendable future-directed lightlike geodesic ray through p intersects S.*

Proof. Let $p \in AdS_3$ be a point such that every past-directed lightlike geodesic ray through p intersects the surface S. Then, every past-directed lightlike geodesic ray through p intersects (transversally) the surface S at exactly one point (see Proposition 5.10). Hence, the set C of all the points of S that can be joined from p by a past-directed lightlike geodesic ray is homeomorphic to a circle. Therefore, C is the boundary of a closed disk $D \subset S$ (recall that S is a properly embedded disc, see Proposition 5.1). Let L be the union of all the segments of lightlike geodesics joining p to a point of C. The union of D and L is a non-pathological sphere. By Jordan-Schoenflies theorem, this topological sphere is the boundary of a ball $B \subset AdS_3$. A non-spacelike curve cannot escape B through L; as a consequence, every past-inextendable non-spacelike curve through p must escape from B through D ; in particular, every past-inextendable non-spacelike curve through p must intersect S. Hence, the point p is in $D^+(S)$. ☐

Remark 5.15. *Since the surface Σ_0 is a Cauchy surface in M, the range $\mathcal{D}(\widetilde{M})$ of the developping map $\mathcal{D}$ must be contained in the Cauchy development of the surface $S_0 = \mathcal{D}(\widetilde{\Sigma}_0)$.*

We now define another domain, the *black domain* $E(\partial S_0)$, which, as we will prove later, coincides with the Cauchy development $D(S)$.

Definition of the set $E(\partial S_0)$. The set

$$E(\partial S_0) = \{r \in \mathbb{S}^3 \mid B_Q(r,q) < 0 \text{ for every } q \in \partial S_0\}$$

is called the *black domain* of the curve ∂S_0 (explanations on this terminology are provided below).

Remark 5.16. *Here are a few observations about the definition of the set $E(\partial S_0)$:*

(i) *We will prove below (Proposition 6.11)) that the black domain $E(\partial S_0)$ (which is defined above as a subset of the sphere $\mathbb{S}^3$) is actually contained in the anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}_3$. Moreover, we will prove that, for a suitable choice of the point p_0, the set $E(\partial S_0)$ is contained in the affine domain $\mathcal{A}_{p_0}$ (Proposition 6.14).*

(ii) *Consider a point $p_0 \in AdS_3$ such that $E(\partial S_0)$ is contained in $\mathcal{A}_{p_0}$. According to Proposition 4.19, the set $E(\partial S_0)$ is made of the points $r \in \mathcal{A}_{p_0}$ such that there does not exist any causal curve joining r to the curve ∂S_0 within $\mathcal{A}_{p_0}$. In other words, $E(\partial S_0)$ is the set of "all the points of $\mathcal{A}_{p_0}$ that cannot be seen from any point of the curve ∂S_0". This is the reason why we call $E(\partial S_0)$ the* black domain *of the curve ∂S_0.*

(iii) *The black domain $E(\partial S_0)$ is clearly a convex subset of $\mathbb{S}^3$ (by construction, it is an intersection of convex subsets of $\mathbb{S}^3$). In particular, $E(\partial S_0)$ is connected.*

(iv) *Here is a nice way to visualize $E(\partial S_0)$. Consider a point $p_0 \in \mathbb{A}\mathrm{d}\mathbb{S}_3$ such that $E(\partial S_0)$ is contained in the affine domain $\mathcal{A}_{p_0}$ (see Proposition 6.14). Using the diffeomorphism Φ_{p_0}, we can identify $\mathcal{A}_{p_0}$, $\partial\mathcal{A}_{p_0}$, ∂S_0, $E(\partial S_0)$ with some subsets of $\mathbb{R}^3$ (in particular, $\partial\mathcal{A}_{p_0}$ is identified with the hyperboloid of equation $(x^2 + y^2 - z^2 = 1)$). Given $q \in \partial S_0$, the set $T_q = \{r \in \mathcal{A}_p \mid B_Q(q,r) = 0\}$ is the affine plane of $\mathbb{R}^3$ which is tangent to the hyperboloid $\partial\mathcal{A}_{p_0}$ at q. If we define the set $E_q = \{r \in \mathcal{A}_p \mid B_Q(q,r) < 0\}$ as the connected component of $\mathbb{R}^3 \setminus T_q$ containing at least* *one point of ∂S_0, ∂S_0 is contained in the closure of E_q, and the set $E(\partial S_0)$ is the intersection over all $q \in \partial S_0$, of the E_q's.*

(v) *Let r be a point on the boundary (in $\mathbb{A}\mathrm{d}\mathbb{S}_3$) of $E(\partial S_0)$. The definition of the set $E(\partial S_0)$ and the compactness of the curve ∂S_0 imply that we have $B_Q(r, q) = 0$ for some point q on the curve ∂S_0. Hence, by Remark 4.12, there exists a lightlike geodesic γ passing through r, such that one of the two ends of γ is a point of the curve ∂S_0.*

Proposition 5.17. *The surface S_0 is contained in $E(\partial S_0)$.*

Proof. Let p be a point in S_0. By Proposition 5.11, the surface $\overline{S_0}$ is contained in the affine domain $\mathcal{A}_p \cup \partial\mathcal{A}_p$. By Proposition 4.19, if for some q in ∂S_0 we have $B_Q(p, q) \geq 0$, there is a causal curve in $\mathcal{A}_p$ joining p to q. But such a curve cannot exist according to Proposition 5.12. The proposition follows. □

Proposition 5.18. *The black domain $E(\partial S_0)$ contains the Cauchy development $D(S_0)$.*

Proof. Assume the contrary. Since $D(S_0)$ and $E(\partial S_0)$ have a non-empty intersection (the surface S_0 is contained in both $D(S_0)$ and $E(\partial S_0)$), and since $D(S_0)$ is connected, $D(S_0)$ must contain some point r of the boundary of $E(\partial S_0)$. By item (v) of Remark 5.16, there exists a lightlike geodesic γ passing through r, such that one of the ends of γ is a point q on the curve ∂S_0. Since r is in $D(S_0)$, the lightlike geodesic γ must intersect the surface S_0. But, this is impossible according to Proposition 5.10. □

Corollary 5.19. *The black domain $E(\partial S_0)$ and the Cauchy development $D(S_0)$ do not contain any timelike geodesic.*

Proof. Let γ be a timelike geodesic. Recall that γ is a closed geodesic. Consider all future oriented lightlike geodesic rays starting from a point of γ: the union of their future extremities covers the whole $\partial\mathbb{A}\mathrm{d}\mathbb{S}_3$, in particular, it contains ∂S_0. It follows that γ cannot be contained in the black domain $E(\partial S_0)$. Therefore, the corollary follows from Proposition 5.18. □

Proposition 5.20. *The developping map $\mathcal{D} : \widetilde{M} \to AdS_3$ is one-to-one.*

Proof. Consider the lifting $\tau : \widetilde{M} \to \mathbb{R}$ of any time function on M. Select any timelike geodesic Δ_0 of AdS_3. According to the corollary 5.19, the intersection between Δ_0 and $E(\partial S_0)$ is a subarc $I \approx \mathbb{R}$ (it is connected since $E(\partial S_0)$ is convex). Every level set of τ is the lift of a Cauchy surface of M. So, by Proposition 5.1 and Remark 5.2, for every t in $\mathbb{R}$, the image of $\tau^{-1}(t)$ under $\mathcal{D}$ is a spacelike surface that intersects Δ_0 at one and only one point $d(t)$. Clearly, d is a strictly increasing function, hence, it is injective. Therefore, for any p and q in $\widetilde{M}$, if $\mathcal{D}(p) = \mathcal{D}(q)$, then $\tau(p) = \tau(q)$: p and q belongs to the same spacelike level set of τ. According to (the proof of) proposition 5.1, the restriction of $\mathcal{D}$ to every level of τ is injective. Hence, $p = q$. □

Proposition 5.21. *The holonomy group $\Gamma = \rho(\pi_1(M))$ acts freely, and properly discontinuously on the Cauchy development $D(S_0)$ of the surface S_0.*

Proof. First note that the group Γ acts freely and properly discontinuously on the surface $S_0 = \mathcal{D}(\widetilde{\Sigma}_0)$ (since $\mathcal{D} : \widetilde{\Sigma}_0 \to AdS_3$ is a proper embedding).

Suppose that the group Γ does not act freely on the future Cauchy development $D^+(S_0)$. Then, there exists an element γ of Γ which fixes a point p of $D^+(S_0)$. Then, as in the proof of Lemma 5.14, we consider the set C of all the points of S_0 that can be joined from p by a past-directed lightlike geodesic ray. The set C is homeomorphic to a circle, and thus, it is the boundary of a closed disc $D \subset S_0$. The disc D must be invariant under γ (since the surface S_0 is Γ-invariant, and since γ fixes the point p). Hence, by Brouwer's theorem, γ fixes a point in D.

In particular, γ fixes a point in S_0. This contradicts the fact that Γ acts freely on S_0. Hence, Γ must act freely on $D^+(S_0)$. The same arguments show that Γ acts freely on $D^-(S_0)$.

Now, let K be a compact subset contained in $D^+(S_0)$. All the points of intersection of the past-directed lightlike geodesic rays emanating from the points of K with the surface S_0 belong to some compact subset K' of the surface S_0. Since Γ maps lightlike geodesic rays to lightlike geodesic rays, the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is contained in the set $\{\gamma \in \Gamma \mid \gamma K' \cap K' \neq \emptyset\}$. Hence, the proper discontinuity of the action of Γ on $D^+(S_0)$ follows from the proper discontinuity of the action on S_0. The same arguments show that Γ acts properly discontinuously on $D^-(S_0)$. $\square$

Proposition 5.22. *The spacetime M is isometric to the quotient $\Gamma\backslash D(S_0)$ (the isometry being induced by the developping map $\mathcal{D}$).*

Proof. By Proposition 5.21, the quotient $\Gamma\backslash D(S_0)$ is a manifold (which is automatically a globally hyperbolic, since it is the quotient of the Cauchy development $D(S_0)$). By Remark 5.15 and Proposition 5.20, the developping map $\mathcal{D}$ induces an isometric embedding of M in $\Gamma\backslash D(S_0)$. Since M is assumed to be maximal as a globally hyperbolic manifold, this embedding must be onto. $\square$

According to Proposition 5.22, constructing a surface in M with some specified geometrical properties amounts to constructing a Γ-invariant surface in $D(S_0)$. In particular, we will use the following remark several times:

Remark 5.23. *If S is a Γ-invariant spacelike surface contained in the Cauchy development $D(S_0)$, then $\Gamma\backslash S$ is a Cauchy surface in $M = \Gamma\backslash D(S_0)$. Indeed, $\Gamma\backslash S$ is a spacelike compact surface in $M = \Gamma\backslash D(S_0)$, and every compact spacelike surface in M is a Cauchy surface.*

6 Proof of Theorem 1.1 in the case $g \geq 2$

We have to prove that M admits a CMC time function. In this section, we give the proof in the case $g \geq 2$; the proof in the other case $g = 1$ (see Remark 5.3) is completely different and will be achieved in section 7.

In subsection 6.1, we will explain why in the case $g \geq 2$, this problem reduces to the proof of the existence of a pair of barriers in M.

In subsection 6.2, we prove that when Σ has higher genus, then the compactified surface $\overline{S_0}$ is strictly achronal. In subsection 6.3, we study the intersection C_0 of $\mathbb{A}d\mathbb{S}_3$ with the convex hull of the curve ∂S_0. In particular, we prove that C_0 is contained in the Cauchy development $D(S_0)$, so that we may consider the projection $\Gamma\backslash C_0$ of C_0 in $\Gamma\backslash D(S_0) \simeq M$. We also complete the study in the previous section above by proving, for example, that the Cauchy development and the black domain coincide[3].

In subsection 6.4, we define the notion of convexity and concavity for spacelike surfaces in $\mathbb{A}d\mathbb{S}_3$, and we prove that the boundary of $C(S_0)$ in $\mathbb{A}d\mathbb{S}_3$ is the union of two disjoint spacelike topological surfaces S_0^- and S_0^+, repectively convex and concave. The projections $\Sigma_0^- = \Gamma\backslash S_0^-$ and $\Sigma_0^+ = \Gamma\backslash S_0^+$ of these surfaces in $\Gamma\backslash D(S_0) \simeq M$ is "almost a pair of barriers". There are still two small problems: in general, the surfaces Σ_0^- and Σ_0^+ have totally geodesic regions (whereas, for barriers, we need surfaces with positive and negative mean curvature), and in general, these are only topological surfaces (whereas, for barriers, we need surfaces of class C^2). The purpose of subsections 6.5 and 6.6 is to approximate the surfaces Σ_0^- and Σ_0^+ by a true pair of barriers.

[3]This last statement remains true in the case $g = 1$, but the proof is quite different than those of the case $g \geq 2$.

6.1 Reduction of Theorem 1.1 to the existence of a pair of barriers

V. Moncrief has proved that the solutions of the vacuum Einstein equation in dimension $2+1$ with a compact Cauchy surface can be described as the orbits of a non-autonomous hamiltonian flow on a finite-dimensional space (namely the cotangent bundle of the Teichmüller space of the Cauchy surface). Using this hamiltonian flow, L. Andersson, Moncrief and A. Tromba have obtained the following theorem ([4, corollary 7]):

Theorem 6.1 (Andersson, Moncrief, Tromba). *Let N be a 3-dimensional maximal globally hyperbolic spacetime, with constant curvature, and with closed Cauchy surfaces of genus $g \geq 2$. If N admits a CMC Cauchy surface, then it admits a CMC time function.*

Thanks to Theorem 6.1, the proof of Theorem 1.1 is reduced to the proof of the existence of a CMC Cauchy surface. The existence of CMC surfaces, in particular the existence of surfaces with zero mean curvature, has been studied in many contexts. The problem usually splits into two disjoint steps : a geometrical step which consists in constructing some surfaces with (non-constant) negative and positive mean curvature called *barriers*, and an analytical step which consists in solving the appropriate PDE to prove the existence of a surface with zero mean curvature assuming the existence of barriers. In our context, the needed statement for the second step is due to C. Gerhardt (see [11, Theorem 6.1][4]):

Definition 6.2. *A* pair of barriers *in a three-dimensional globally hyperbolic Lorentzian manifold N is a pair of disjoint Cauchy surfaces Σ^- and Σ^+ in N, such that Σ^+ is in the future of Σ^-, the supremum of the mean curvature of Σ^- is negative, and the infimum of the mean curvature of Σ^+ is positive.*

Theorem 6.3 (Gerhardt). *Let N be a three-dimensional globally hyperbolic Lorentzian manifold, with compact Cauchy surfaces. Assume that there exists a pair of barriers in N. Then, N admits a Cauchy surface with zero mean curvature in N (*i.e., *a maximal Cauchy surface).*

Using the results of Andersson-Moncrief-Tromba and Gerhardt stated above, the proof of our main theorem reduces to the proof of the existence of a pair of barriers in M.

6.2 Strict achronality

Proposition 6.4. *The topological surface $\overline{S_0}$ is spacelike.*

Remark 6.5. *This Proposition is false without the assumption that the Cauchy surface Σ_0 has genus $g \geq 2$, see Remark 7.6.*

Proof. We already know that $\overline{S_0}$ is non-timelike, and that S_0 is spacelike. Hence, $\overline{S_0}$ is spacelike if and only if the curve ∂S_0 does not contain any non-trivial lightlike arc. Therefore, $\overline{S_0}$ is spacelike if and only if ∂S_0 does not contain any non-trivial arc of some leaf of the left or the right ruling of $\partial \mathbb{A}d\mathbb{S}_3$.

Let us denote by $\mathbb{RP}^1_L$ (resp. $\mathbb{RP}^1_R$) the space of the leaves of the left (resp. right) ruling of $\partial \mathbb{A}d\mathbb{S}_3$. We recall that the action of the holonomy ρ on $\mathbb{RP}^1_L$ reduces to the action of ρ_R (since, ρ_L preserves individually each cicrle of the left ruling). Similarly, the action of ρ on $\mathbb{RP}^1_R$ reduces to the action of ρ_L.

Lemma 6.6. *The actions of the representations ρ_L and ρ_R respectively on $\mathbb{RP}^1_R$ and $\mathbb{RP}^1_L$ are minimal.*

[4]The result proved by Gerhardt is actually more general than the statement that we give below.

Proof. Let p be a point of the surface S_0, and n the future-pointing unitary normal vector of S_0 at p. If v is a unitary vector tangent to S_0 at p, then $n+v$ is a future pointing lightlike vector. The lightlike geodesic directed by $n+v$ is tangent to ∂AdS_3 at two antipodal points (Remark 4.11). These two antipodal points lie on the same leaf of the right ruling; denote by $R_{[\lambda:\mu]}$ this leaf (with $[\lambda:\mu] \in \mathbb{RP}^1_L$). The map $(p,v) \to (p, R_{[\lambda:\mu]})$ identifies the unitary tangent bundle of the surface Σ_0 with the flat $\mathbb{RP}^1$ bundle over Σ_0 given by $\pi_1(\Sigma_0)\backslash(S_0 \times \mathbb{RP}^1)$ where $\gamma \in \pi_1(M) = \pi_1(\Sigma_0)$ acts by $\gamma.(p,[\lambda:\mu]) = (\rho(\gamma)(p), \rho_L(\gamma)([\lambda:\mu]))$. Hence, the Euler class of the representation ρ_L is the Euler class of the unitary tangent bundle of Σ_0. By a theorem of Goldman (see [12])[5], this implies $\rho_L(\pi_1(M))$ is a cocompact Fuchsian subgroup of $SL(2,\mathbb{R}) \times Id \simeq SL(2,\mathbb{R})$. In particular, the action of ρ_L on $\mathbb{RP}^1_R$ is minimal. □

End of the proof of Proposition 6.4. Denote by U the open subset of ∂S_0, defined as the union of the interiors of all the non-trivial arcs of leaves of left ruling contained in ∂S_0. Note that the holonomy ρ preserves the open set U. Now, let $U_R \subset \mathbb{RP}^1_R$ be the set of all leaves of the right ruling that intersect U. Then U_R is an open subset of $\mathbb{RP}^1_R$ which is preserved by ρ_L. Hence, U_R is either empty or equal to $\mathbb{RP}^1_R$. But the equality $U_R = \mathbb{RP}^1_R$ would imply that ∂S_0 is a leaf of the left ruling, which is impossible by Proposition 5.11. Hence, U_R is empty, *i.e.* the curve ∂S_0 does not contain any non-trivial arc of leaf of the left ruling. Similarly, for the right ruling. This completes the proof. □

Remark 6.7. *On the one hand, Proposition 5.1 implies that the action of Γ on the surface S_0 is free and properly discontinous. On the other hand, Lemma 6.6 implies that the action of Γ on ∂S_0 is minimal. As a consequence, the curve ∂S_0 is the limit set of the action of Γ on the surface S_0.*

We thus obtain a more powerfull version of Proposition 5.12:

Corollary 6.8. *For every $p \in \mathbb{A}d\mathbb{S}_3$ such that $\overline{S_0} \subset \mathcal{A}_p \cup \partial\mathcal{A}_p$, the surface $\overline{S_0}$ is a strictly achronal subset of $\mathcal{A}_p \cup \partial\mathcal{A}_p$ (*i.e.* a causal curve contained in $\mathcal{A}_p \cup \partial\mathcal{A}_p$ can not intersect $\overline{S_0}$ at two distinct points).*

6.3 The convex hull of the curve ∂S_0

In this subsection, we will consider the convex hull $\mathrm{Conv}(\partial S_0)$ of the curve ∂S_0. The main goal is to prove that the set $\mathrm{Conv}(\partial S_0) \setminus \partial S_0$ is contained in the Cauchy development of the surface S_0. We will also prove that the black domain and the Cauchy development coincide.

Definition of the set C_0. Denote by $\mathrm{Conv}(\partial S_0)$ the convex hull in $\mathbb{S}^3$ of the curve ∂S_0 (see subsection 4.4), and consider the set

$$C_0 = \mathrm{Conv}(\partial S_0) \cap \mathbb{A}d\mathbb{S}_3$$

Proposition 6.9. *The set Conv$(\partial S_0) \setminus \partial S_0$ is contained in $E(\partial S_0)$.*

Proof. Let q be a point $\mathrm{Conv}(\partial S_0) \setminus \partial S_0$, and let $\widehat{q}$ be any point in $\pi^{-1}(\{q\})$ (recall that π is the radial projection of $\mathbb{R}^4 \setminus \{0\}$ on $\mathbb{S}^3$). Let r be a point in ∂S_0, and let $\widehat{r}$ be any point in $\pi^{-1}(\{r\})$. We have to prove that $B_Q(q,r)$ is negative, *i.e.* that $B_Q(\hat{q},\hat{r})$ is negative. Since $\widehat{q}$ is in $\pi^{-1}(\mathrm{Conv}(\partial S_0))$, one can find points $\widehat{q}_1,\dots,\widehat{q}_n \in \pi^{-1}(\partial S_0)$, and positive numbers $\alpha_1,\dots,\alpha_n$, such that $\alpha_1+\dots+\alpha_n = 1$, and such that $\widehat{q} = \alpha_1\widehat{q}_1 + \dots + \alpha_n\widehat{q}_n$. We denote by $q_1,\dots,q_n$ the projections of the points $\widehat{q}_1,\dots,\widehat{q}_n$. For each $i \in \{1,\dots,n\}$, there are two possibities:
— either $q_i = r$, and then we have $B_Q(\widehat{q}_i,\widehat{r}) = B_Q(\widehat{r},\widehat{r}) = 0$ (since $\widehat{r}$ is on the quadric $(Q=0)$),

[5] Here, we use the fact that the genus of Σ_0 is at least 2.

— or $q_i \neq r$, and then corollary 6.8 and Proposition 4.19 imply that $B_Q(\widehat{q_i}, \widehat{r})$ is negative. Moreover, at least one q_i's is different from r (otherwise, we would have $q_1 = \cdots = q_n = q$, which is absurd since q is not on ∂S_0). Hence, the quantity $B_Q(\widehat{q}, \widehat{r}) = \alpha_1 B_Q(\widehat{q_1}, \widehat{r}) + \cdots + \alpha_n B_Q(\widehat{q_n}, \widehat{r})$ is negative. The proposition follows. □

Lemma 6.10. *For every point $q \in \partial\mathbb{A}d\mathbb{S}_3$, there exists a point $r \in \partial S_0$, such that $B_Q(q,r)$ is non-negative. Moreover, if the point q is not on the curve ∂S_0, then the point r can be choosen such that $B_Q(q,r)$ is positive.*

Proof. Let q be a point in $\partial\mathbb{A}d\mathbb{S}_3$. Denote by $[x_1 : x_2 : x_3 : x_4]$ the coordinates of q in $\mathbb{S}^3$. Remark 5.8 imply that there exists x_1', x_2' such that the point r of coordinates $[x_1' : x_2', x_3, x_4]$ is on the curve ∂S_0. The sign of $B_Q(q,r)$ is the sign of the expression $-x_1x_1' - x_2x_2' + x_3^2 + x_4^2$ (we recall that only the sign of $B_Q(q,r)$ is well-defined, see Remark 4.8). Since the points q and r are both on $\partial\mathbb{A}d\mathbb{S}_3$, we have $Q([x_1 : x_2 : x_3 : x_4]) = Q([x_1' : x_2' : x_3 : x_4]) = 0$. Hence, we have $-x_1x_1' - x_2x_2' + x_3^2 + x_4^2 = \frac{1}{2}((x_1 - x_1')^2 + (x_2 - x_2')^2)$. As a consequence, $B_Q(q,r)$ is non-negative. Moreover, if q is not on the curve ∂S_0, then (x_1, x_2) is different from (x_1', x_2'), and thus, $B_Q(q,r)$ is positive. □

Corollary 6.11. *The black domain $E(\partial S_0)$ is contained in $\mathbb{A}d\mathbb{S}_3$.*

Proof. Lemma 6.10 says that the intersection of $\partial\mathbb{A}d\mathbb{S}_3$ with $E(\partial S_0)$ is empty. Since $E(\partial S_0)$ is connected, this implies that $E(\partial S_0)$ is either contained in $\mathbb{A}d\mathbb{S}_3$, or disjoint from $\mathbb{A}d\mathbb{S}_3$. But, the intersection of $E(\partial S_0)$ with $\mathbb{A}d\mathbb{S}_3$ is non-empty (by Proposition 6.9, for example). Hence, $E(\partial S_0)$ is contained in $\mathbb{A}d\mathbb{S}_3$. □

Corollary 6.12. *The set $Conv(\partial S_0) \setminus \partial S_0$ is contained in $\mathbb{A}d\mathbb{S}_3$, i.e., $C_0 = Conv(\partial S_0) \setminus \partial S_0$.*

Proof. The corollary follows immediately from Proposition 6.9 and corollary 6.11. □

We will denote by $\overline{E(\partial S_0)}$ the closure of the black domain $E(\partial S_0)$ in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$.

Corollary 6.13. *The intersection of $\overline{E(\partial S_0)}$ with $\partial\mathbb{A}d\mathbb{S}_3$ is the curve ∂S_0.*

Proof. Proposition 6.9 implies that every point of the curve ∂S_0 is in $\overline{E(\partial S_0)}$. Conversely, let q be a point in $\partial\mathbb{A}d\mathbb{S}_3 \setminus \partial S_0$. According to Lemma 6.10, there exists a point $r \in \partial S_0$ such that $B_Q(q,r) > 0$. By continuity of the bilinear form B_Q, there exists a neighbourhood U of q in $\mathbb{S}^3$, such that $B_Q(q',r) > 0$ for every $q' \in U$. In particular, there exists a neighbourhood U of q which is disjoint from $E(\partial S_0)$. Hence, q is not in $\overline{E(\partial S_0)}$. □

Proposition 6.14. *There exists a point $p_0 \in \mathbb{A}d\mathbb{S}_3$ such that $E(\partial S_0)$ is contained in the affine domain $\mathcal{A}_{p_0}$.*

Addendum. *If the curve ∂S_0 is not flat*[6]*, then one can choose the point p_0 such that $\overline{E(\partial S_0)}$ is contained in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$.*

Lemma 6.15. *For every point $p \in C(\partial S_0) = Conv(\partial S_0) \setminus \partial S_0$, the black domain $E(\partial S_0)$ is disjoint from the totally geodesic surface p^* (and thus, is disjoint from the closed surface $\overline{p^*}$).*

Proof. Let p be a point in $C(\partial S_0)$, and $\hat{p}$ be a point in $\mathbb{R}^4 \setminus \{0\}$ such that $\pi(\hat{p}) = p$. Since p is in Conv(∂S_0), one can find some points $\widehat{p}_1, \ldots, \widehat{p}_n \in \pi^{-1}(\partial S_0)$ and some positive numbers $\alpha_1, \ldots, \alpha_n$ such that $\widehat{p} = \alpha_1 \widehat{p}_1 + \cdots + \alpha_n \widehat{p}_n$. Let q be a point in $E(\partial S_0)$ and $\hat{q}$ be a point in $\mathbb{R}^4 \setminus \{0\}$ such that $\pi(\widehat{q}) = q$. Since q is in $E(\partial S_0)$, the quantity $B_Q(\widehat{p_i}, \widehat{q})$ is negative for every

[6] We say that the curve ∂S_0 is *flat* if it is the boundary of a totally geodesic subspace of $\mathbb{A}d\mathbb{S}_3$, or equivalently, if it is contained in a great 2-sphere in $\mathbb{S}^3$.

i. Hence, the quantity $B_Q(\widehat{p},\widehat{q}) = \alpha_1 B_Q(\widehat{p}_1,\widehat{q}) + \cdots + \alpha_n B_Q(\widehat{p}_n,\widehat{q})$ is negative. In particular, the point q is not on the surface $p^* = \{r \in \mathbb{A}d\mathbb{S}_3 \mid B_Q(\widehat{p},\widehat{r}) = 0\}$. This proves that set $E(\partial S_0)$ is disjoint from the totally geodesic surface p^*. Since $E(\partial S_0)$ is contained in $\mathbb{A}d\mathbb{S}_3$, it is also disjoint from the closed surface $\overline{p^*}$. □

Proof of Proposition 6.14. Let p_0 be a point in C_0. By Lemma 6.15, $E(\partial S_0)$ is disjoint from the totally geodesic surface p_0^*. Since $E(\partial S_0)$ is connected, this implies that $E(\partial S_0)$ is contained in one of the two connected components of $\mathbb{A}d\mathbb{S}_3 \setminus p_0^*$. By Proposition 6.9, the point p_0 is in $E(\partial S_0)$. Hence, $E(\partial S_0)$ is contained in the connected component of $\mathbb{A}d\mathbb{S}_3 \setminus p_0^*$ containing p_0, that is, in $\mathcal{A}_{p_0}$. □

Proof of the addendum. If ∂S_0 is not flat, then the set C_0 has non-empty interior. Let p_0 be a point in the interior of $C(\partial S_0)$. On the one hand, the set $E(\partial S_0)$ is disjoint from the closed surface $\overline{p^*}$ for every $p \in C_0$. On the other hand, the union of all the surfaces $\overline{p^*}$ when p ranges over C_0 is a neighbourhood (in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$) of the surface $\overline{p_0^*}$. Hence, $E(\partial S_0)$ is disjoint from a neighbourhood of the surface $\overline{p_0^*}$. Hence, $\overline{E(\partial S_0)}$ is disjoint from the surface $\overline{p_0^*}$. Moreover, by Proposition 6.9, the point p_0 is in $\overline{E(\partial S_0)}$. Therefore, $\overline{E(\partial S_0)}$ is contained in the connected component of $(\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3) \setminus \overline{p_0^*}$ containing p_0, *i.e.* is contained in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$. □

From now on, we fix a point $p_0 \in \mathbb{A}d\mathbb{S}_3$, such that $\overline{E(\partial S_0)}$ is contained in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$.

Proposition 6.16. *The black domain $E(\partial S_0)$ coincides with the Cauchy development $D(S_0)$.*

Proof. Proposition 5.18 provides an inclusion. To prove the other inclusion, we work in the affine domain $\mathcal{A}_{p_0}$. Let p be a point in $E(\partial S_0)$. By Remark 5.13, every point of $\mathcal{A}_{p_0}$ is either in the past, or in the future of the surface S_0. We assume, for example, that p is in the future of S_0. We will prove that p is in $D^+(S_0)$. For that purpose, we consider a past-directed lightlike geodesic ray γ emanating from p, and we denote by q the past end of γ.

Claim. The geodesic ray γ intersects the boundary of $E(\partial S_0)$ at some point r in the past of S_0.

To prove this claim, we argue by contradiction. First, we suppose that the geodesic ray γ is contained in $E(\partial S_0)$. Then, by Proposition 6.11 and corollary 6.13, the past end of γ must be a point q of the curve ∂S_0. But then, we have $B_Q(p,q) = 0$, and this contradicts the fact that p is in $E(\partial S_0)$. Now, we suppose that the geodesic ray γ intersects the boundary $E(\partial S_0)$ at some point r in the future of the surface S_0. By item (v) of Remark 5.16, there exists a lightlike geodesic ray γ' emanating from r, such that the end of γ' is a point q of the curve ∂S_0. The geodesic ray γ' must be past-directed from r to q, since r is in the future of the surface S_0. So, we have a past-directed lightlike geodesic segment going from p to r, and a past-directed geodesic ray going from r to q; concatenating these two curves, we obtain a piecewise C^1 causal curve going from p to $q \in \partial S_0$. This contradicts the fact that p is in $E(\partial S_0)$ (see item (ii) of Remark 5.16) and completes the proof of the claim.

Since the point p is in the future of the surface S_0, and since the point r given by the claim is in the past of the surface S_0, the geodesic ray γ must intersect the surface S_0. So, we have proved that every past-directed geodesic ray emanating from p intersects the surface S_0. Hence, the point p is in $D^+(S_0)$ (Lemma 5.14). This proves that $E(\partial S_0)$ is contained in $D(S_0)$. □

Remark 6.17. *Proposition 6.16 implies in particular that the Cauchy development $D(S_0)$ depends only on the curve ∂S_0,* i.e. *if S is another complete spacelike surface in $\mathbb{A}d\mathbb{S}_3$ such that $\partial S = \partial S_0$, then $D(S) = D(S_0)$.*

Remark 6.18. *Let Σ be any Cauchy surface in M, and let $S := \mathcal{D}(\widetilde{\Sigma})$. On the one hand, we have $D(S) = D(S_0) = \mathcal{D}(\widetilde{M})$. On the other hand, Propositions 6.13 and 6.16 imply that the curve ∂S_0 is the intersection of the closure in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ of $D(S_0)$ with $\partial\mathbb{A}d\mathbb{S}_3$. Similarly,* *the curve ∂S is the intersection of the closure in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$ of $D(S)$ with $\partial\mathbb{A}d\mathbb{S}_3$. As a consequence, we have $\partial S = \partial S_0$.*

Remark 6.19. *For every point $p \in D(S_0) = \mathcal{D}(\widetilde{M})$, one can find a Cauchy surface Σ in M such that $p \in \mathcal{D}(\widetilde{\Sigma})$. By Remark 6.7 and 6.18, the limit set of the action of Γ on the surface S is the curve $\partial S = \partial S_0$. As a consequence, the limit set of the action of Γ on $D(S_0)$ is also the curve ∂S_0.*

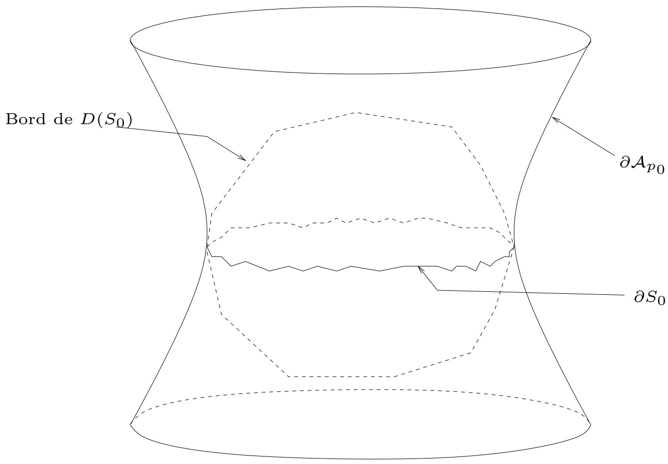

Figure 1: The affine domain $\mathcal{A}_{p_0}$, the curve ∂S_0 and the Cauchy development $D(S_0)$.

Interlude: proof of Theorem 1.1 in the case where ∂S_0 is flat

Our strategy for proving the existence of a pair of barriers in M does not work in the particular case where ∂S_0 is flat, mostly because the addendum of Proposition 6.14 is false when ∂S_0 is flat. This is not a big problem, since there is a direct and very short proof of Theorem 1.1 in this particular case:

Proof of Theorem 1.1 in the case where ∂S_0 is flat. Assume that ∂S_0 is flat. Then it is the boundary of a totally geodesic subspace P_0 of $\mathbb{A}d\mathbb{S}_3$. This totally geodesic subspace is necessarly spacelike, since the curve ∂S_0 is spacelike. By construction, P_0 is contained in C_0; hence, it is contained in the Cauchy development $D(S_0)$ (Proposition 6.16 and 6.9). Moreover, the holonomy group $\Gamma = \rho(\pi_1(M))$ preserves P_0 (since it preserves the curve ∂S_0). As a consequence, $\Gamma\backslash P_0$ is a totally geodesic compact spacelike surface in $\Gamma\backslash D(S_0) \simeq M$. In particular, $\Gamma\backslash P_0$ is a Cauchy surface with zero mean curvature in M. Applying Theorem 6.1, we obtain Theorem 1.1. $\square$

Assumption. From now on, we assume that the curve ∂S_0 is not flat.

6.4 A pair of convex/concave topological Cauchy surfaces

In this subsection, we will first define some notions of convexity and concavity for spacelike surfaces in M. The main interesting feature of this notion for our purpose is the fact that the mean curvature of a smooth convex (resp. concave) spacelike surface is always non-positive (resp. non-negative). Then, we will exhibit a pair of disjoint topological Cauchy surfaces (Σ_0^-, Σ_0^+) in M, such that Σ_0^- is convex, Σ_0^+ is concave, and Σ_0^+ is in the future of Σ_0^-.

6.4.1 Convex and concave surfaces in $\mathbb{A}d\mathbb{S}_3$

Let S be a topological surface in $\mathcal{A}_{p_0}$, and q be a point of S. A *support plane* of S at q is a (2-dimensional) totally geodesic subspace[7] P of $\mathcal{A}_{p_0}$, such that $q \in P$, and such that S is contained in the closure of one of the connected components of $\mathcal{A}_{p_0} \setminus P$.

Remark 6.20. *Let S be a topological surface in $\mathcal{A}_{p_0}$. If S is spacelike (in the sense of definition 5.9), then every support plane of S is spacelike. Conversely, if S admits a spacelike support plane at every point, then S is spacelike.*

Remark 6.21. *Let S be a topological surface in $\mathcal{A}_{p_0}$ and P be a spacelike support plane of S. Then, S is contained in the causal past[8] of P, or S is contained in the future of P (see Remark 4.20).*

Let S be a topological spacelike surface in $\mathcal{A}_{p_0}$. We say that S is *convex*, if it admits a support plane at each of its points, and if it is contained in the future of all its support planes. We say that S is *concave*, if it admits a support plane a each of its points, and if it is contained in the past of all its support planes.

Now, let Σ be a topological spacelike surface in M, $\widetilde{\Sigma}$ be a lift of Σ in $\widetilde{M}$, and $S = \mathcal{D}(\widetilde{\Sigma})$. Note that S is a topological spacelike surface contained in $\mathcal{D}(\widetilde{M}) \subset \mathcal{A}_{p_0}$ (see section 5.2). We say that Σ is *convex* (resp. *concave*) if S is convex (resp. concave).

Proposition 6.22. *Let Σ be a C^2 spacelike surface in M. If Σ is convex, then Σ has non-positive mean curvature. If Σ is concave, then Σ has non-negative mean curvature.*

Proof. Let $\widetilde{\Sigma}$ be a lift of Σ in $\widetilde{M}$, and let $S = \mathcal{D}(\widetilde{M})$. Assume that Σ is convex. Then S is convex. Hence, for every $q \in S$, the surface S admits a spacelike support plane P_q at q, and is contained in the future of P_q. By Lemma 2.3, the mean curvature of the surface S at q is smaller or equal than the mean curvature of the support plane P_q. But, since P_q is totally geodesic, it has zero mean curvature. Hence, the surface S has non-positive mean curvature. Hence, the surface Σ also has non-positive mean curvature (since the developping map $\mathcal{D}$ is locally isometric). □

The notions of convexity and concavity defined above can only help us in finding spacelike surfaces with non-positive (resp. non-positive) mean curvature. Yet, to apply Gerhardt's Theorem 6.3, we need to find spacelike surfaces with positive (resp. negative) mean curvature. This is the reason why we will define below a notion of *uniformly curved surface* in M.

Let S be a topological surface in $\mathbb{R}^3$, and q be a point on S. We fix a Euclidean metric on $\mathbb{R}^3$. We say that the surface S is *more curved than a sphere of radius R at q*, if there exists a closed Euclidean ball B of radius R, such that q is on the boundary of B, and such that B contains a neighbourhood of q in S.

Remark 6.23. *Assume that the surface S is C^2. Then, S is more curved than a sphere of radius R at q if and only if the osculating quadric of S at q is an ellipsoid of diameter smaller than $2R$.*

Consider a topological surface Σ in M, and a lift $\widetilde{\Sigma}$ of Σ. Let $S = \mathcal{D}(\widetilde{\Sigma})$. We see Σ as a surface in $\mathbb{R}^3$. Let $\Delta \subset \widetilde{\Sigma}$ be a fundamental domain of the covering $\widetilde{\Sigma} \to \Sigma$, and let $D = \mathcal{D}(\Delta)$. We say that the surface Σ is *uniformly curved*, if there exists $R \in (0, +\infty)$ such that the surface S is more curved than a sphere of radius R at each point of D. It is easy to verify that this definition depends neither on the choice of the fundamental domain Δ, nor on the choice of

[7] By a *totally geodesic subspace of* $\mathcal{A}_{p_0}$, we mean the intersection of a totally geodesic subspace of $\mathbb{A}d\mathbb{S}_3$ with $\mathcal{A}_{p_0}$. Note that, with this definition, the degenerated totally geodesic subspaces of $\mathcal{A}_{p_0}$ are not connected (although their closure in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$ is connected), but this does not play any role in the subsequent.

[8] By causal past, we mean causal past in $\mathcal{A}_{p_0}$

the Euclidean metric on $\mathbb{R}^3$ (although one has to change the constant R, when changing the fundamental domain Δ or the Euclidean metric on $\mathbb{R}^3$).

Proposition 6.24. *Let Σ be a C^2 spacelike surface in M. If Σ is convex and uniformly curved, then Σ has negative mean curvature. If Σ is concave and uniformly curved, then Σ has positive mean curvature.*

Proof. Let $\widetilde{\Sigma}$ be a lift of Σ in $\widetilde{M}$, and let $S := \mathcal{D}(\widetilde{M})$. Assume that Σ is convex and uniformly curved. Then, S is convex. So, for every $q \in S$, the surface S admits a support plane P_q at q, and is contained in the future of P_q. Moreover, since Σ is uniformly curved, the surface S and the plane P_q do not have the same osculating quadric (see Remark 6.23). By Lemma 2.3, this implies that the mean curvature of S at q is strictly smaller than the mean curvature of the plane P_q. Since P_q is totally geodesic, P_q has zero mean curvature. Hence, S has negative mean curvature. Therefore, Σ also has negative mean curvature. $\square$

6.4.2 Boundary of Γ-invariant convex sets contained in $D(S_0)$

Proposition 6.25. *Let S be a topological surface in $\mathcal{A}_{p_0}$. Assume that S is contained in $D(S_0)$, and that the boundary of S in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$ is equal to the curve ∂S_0. Then every support plane of S is spacelike*[9]*.*

Proof. Using the diffeomorphism Φ_{p_0}, we identify $\mathcal{A}_{p_0}$ with the region of $\mathbb{R}^3$ defined by the inequation $(x^2 + y^2 - z^2 < 1)$, and $\partial\mathcal{A}_{p_0}$ with the one-sheeted hyperboloid $(x^2 + y^2 - z^2 = 1)$. Let q be a point on the surface S and P be a support plane of S at q. The totally geodesic subspace P is the intersection of $\mathcal{A}_{p_0}$ with an affine plane $\widehat{P}$ of $\mathbb{R}^3$.

On the one hand, since P is a support plane of S, the closure of S must be contained in the closure of one of the two connected components of $\mathbb{R}^3 \setminus \widehat{P}$. In particular, the curve ∂S_0 must be contained in the closure of one of the two connected components of $\mathbb{R}^3 \setminus \widehat{P}$. On the other hand, ∂S_0 is a simple closed curve on the hyperboloid $\partial\mathcal{A}_{p_0}$, which is not null-homotopic in $\partial\mathcal{A}_{p_0}$ (see Remark 5.8). Consequently:

Fact 1. The support plane $P = \widehat{P} \cap \mathcal{A}_{p_0}$ does not contain any affine line of $\mathbb{R}^3$. Indeed, if $\widehat{P} \cap \mathcal{A}_{p_0}$ contains an affine line of $\mathbb{R}^3$, then it is easy to see $\widehat{P} \cap \partial\mathcal{A}_{p_0}$ is a hyperbola, and that the two connected components of $\partial\mathcal{A}_{p_0} \setminus \widehat{P}$ are contractible in $\partial\mathcal{A}_{p_0}$ (we recall that $\mathcal{A}_{p_0}$ is the region $(x^2 + y^2 - z^2 < 1)$ in $\mathbb{R}^3$). Hence, every curve contained in the closure of a connected component of $\partial\mathcal{A}_{p_0} \setminus \widehat{P}$ is null-homotopic in $\partial\mathcal{A}_{p_0}$.

Fact 2. If the plane $\widehat{P}$ is tangent to the hyperboloid $\partial\mathcal{A}_p$ at some point r, then r belongs to the curve ∂S_0. Indeed, if $\widehat{P}$ is tangent to the hyperboloid $\partial\mathcal{A}_{p_0}$ at some point r, then every curve contained in the closure of one of the two connected components of $\partial\mathcal{A}_{p_0} \setminus \widehat{P}$ which is not null-homotopic in $\partial\mathcal{A}_{p_0}$ contains r.

Now, we argue by contradiction: we assume that the totally geodesic plane P is not spacelike. Then, P is either timelike (the Lorentzian metric restricted to P has signature $(+,-)$), or degenerated (the Lorentzian metric restricted to P is degenerated). We will show that the two possibilities lead to a contradiction.

– If P is timelike, then P contains timelike geodesics. By Remark 4.15, a timelike geodesic of $\mathcal{A}_{p_0}$ is an affine line of $\mathbb{R}^3$ which is contained in $\mathcal{A}_{p_0}$. Hence, $P = \widehat{P} \cap \mathcal{A}_{p_0}$ contains an affine line of $\mathbb{R}^3$. This is absurd according to Fact 1 above.

– If P is degenerated then P contains lightlike and spacelike geodesic, but does not contain any timelike geodesic. By Remark 4.15, this implies that $\widehat{P}$ is tangent to the hyperboloid $\partial\mathcal{A}_{p_0}$ at some point r. According to Fact 2, the point r must belong to the curve ∂S_0. But then,

[9] Note that, in general, the surface S does not admit any support plane.

Remark 5.16 item (iv) implies that P is disjoint from $E(\partial S_0)$. In particular, the point q is not in $E(\partial S_0)$. This is absurd since, by hypothesis, the surface S is contained in $E(\partial S_0) = D(S_0)$. $\square$

Proposition 6.26. *Let C be a non-empty Γ-invariant closed*[10] *convex subset of $\mathbb{A}d\mathbb{S}_3$, contained in $D(S_0)$. Then:*

(i) *The boundary of C in $\mathbb{A}d\mathbb{S}_3$ is made of two disjoint Γ-invariant topological surfaces S^- and S^+, such that S^- is convex, S^+ is concave, C is in the future of S^- and in the past of S^+.*

(ii) $\Sigma^- := \Gamma\backslash S^-$ *and* $\Sigma^+ := \Gamma\backslash S^+$ *are two disjoint Cauchy surfaces in $\Gamma\backslash D(S_0) \simeq M$. Moreover, Σ^- is convex, Σ^+ is concave, and Σ^+ is in the future of Σ^-. Of course, the boundary of the set $\Gamma\backslash C$ in M is the union of the surfaces Σ^- and Σ^+.*

Proof. Since C is contained in $D(S_0)$, it is also contained in the affine domain $\mathcal{A}_{p_0}$. We denote by ∂C the boundary of C in $\mathcal{A}_{p_0}$, by $\overline{C}$ the closure of C in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$, and by $\overline{\partial C}$ the boundary of $\overline{C}$ in $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$. Of course, we have $\partial C = \overline{\partial C} \cap \mathcal{A}_{p_0} = \overline{\partial C} \setminus \partial\mathcal{A}_{p_0}$.

The set $\overline{C}$ is a compact convex subset of $\mathcal{A}_{p_0} \cup \partial\mathcal{A}_{p_0}$. So, the diffeomorphism Φ_{p_0} maps $\overline{C}$ to a compact convex subset of $\mathbb{R}^3$. Hence, $\overline{\partial C}$ is a Γ-invariant topological sphere. We have to understand the intersection of $\overline{\partial C}$ with $\partial\mathcal{A}_{p_0}$. On the one hand, by hypothesis, C is contained in $D(S_0)$; hence, $\overline{C}$ is contained in $\overline{D(S_0)}$. The intersection of $\overline{D(S_0)}$ with $\partial\mathcal{A}_{p_0}$ is equal to the curve ∂S_0 (see Propositions 6.13 and 6.16). Hence, the intersection of $\overline{\partial C}$ with $\partial\mathcal{A}_{p_0}$ is contained in the curve ∂S_0. On the other hand, C is a non-empty Γ-invariant subset of $D(S_0)$. Hence, the closure of C must contain the curve ∂S_0 (since this curve is the limit set of the action of Γ on $D(S_0)$). As a consequence, we have $\overline{\partial C} \cap \partial\mathbb{A}d\mathbb{S}_3 = \partial S_0$.

We have proved that $\partial C = \overline{\partial C} \setminus \partial\mathcal{A}_{p_0}$ is a Γ-invariant topological sphere minus the Γ-invariant Jordan curve ∂S_0. Hence, ∂C is the union of two disjoint Γ-invariant topological discs S^- and S^+, such that $\partial S^- = \partial S^+ = \partial S_0$. Since the surfaces S^- and S^+ are contained in the boundary of a convex set, they admit a support plane at each of their points. Hence, by Proposition 6.25 and Remark 6.20, the surfaces S^- and S^+ are spacelike. Since S^- is a spacelike disc with $\partial S^- = \partial S_0$, it separates $\mathcal{A}_{p_0}$ into two connected components: the past and the future of S^-. The set C must be contained in one of these two connected components, so C is contained either in the past or in the future of S^-. Similarly, for S^+. Moreover, C can not be in the future (resp. the past) of both S^- and S^+. So, up to exchanging S^- and S^+, the set C is in the future of S^- and in the past of S^+. In particular, S^+ is in the future of S^-. Since C is in the future of S^-, the surface S^- must be in the future of each of its support planes. Hence, the surface S^- is convex. Similar arguments show that S^+ is concave. This completes the proof of (i).

Now, since S^- and S^+ are Γ-invariant spacelike surfaces in $D(S_0)$, their projections $\Sigma^- := \Gamma\backslash S^-$ and $\Sigma^+ := \Gamma\backslash S^+$ are Cauchy surfaces in $\Gamma\backslash D(S_0) \simeq M$ (recall that every compact spacelike surface in M is a Cauchy surface). Of course, Σ^+ is in the future of Σ^-, since S^+ is in the future of S^-. Finally, the convexity of Σ^- and the concavity of Σ^+ follow, by definition, from the convexity of S^- and the concavity of S^-. $\square$

6.4.3 Definition of the topological Cauchy surfaces Σ_0^- and Σ_0^+

The set $C(\partial S_0) = \mathrm{Conv}(\partial S_0) \setminus \partial S_0$ satisfies the hypothesis of Proposition 6.26. Hence, the boundary in $\mathbb{A}d\mathbb{S}_3$ of $C(\partial S_0)$ is made of two disjoint Γ-invariant spacelike topological surfaces S_0^- and S_0^+, such that S_0^- is convex, S_0^+ is concave, and S_0^+ is in the future of S_0^-. Moreover, the surfaces $\Sigma_0^- := \Gamma\backslash S_0^-$ and $\Sigma_0^+ := \Gamma\backslash S_0^+$ are two disjoint topological Cauchy surfaces in $\Gamma\backslash D(S_0) \simeq M$, such that Σ_0^- is convex, Σ_0^+ is concave, and Σ_0^+ is in the future of Σ_0^-.

[10] By such, we mean that C is closed in $\mathbb{A}d\mathbb{S}_3$, but not necessarily in $\mathbb{A}d\mathbb{S}_3 \cup \partial\mathbb{A}d\mathbb{S}_3$. Actually, a non-empty Γ-invariant subset of AdS_3 cannot be closed in $AdS_3 \cup \partial AdS_3$

Definition 6.27. *A pair (S^-, S^+) of disjoint Γ-invariant spacelike topological surfaces in AdS_3 such that S_0^- is convex, S^+ is concave, and S^+ is in the future of S^- is called a* convex trap.

Similarly, a pair (Σ^-, Σ^+) of disjoint spacelike topological surfaces in M such that Σ^- is convex, Σ^+ is concave, and Σ^+ is in the future of Σ^- is called a convex trap.

In both circonstances, a convex trap is uniformly curved if the boundary surfaces S^-, S^+ (or Σ^-, Σ^+)) are uniformly curved. The convex trap is smooth if the boundary surfaces are smooth.

6.5 A pair of uniformly curved convex/concave topological Cauchy surfaces

Our goal is to find a pair of barriers in M. By Proposition 6.24, this goal will be achieved if we find a smooth uniformely curved convex trap. For the moment, the convex trap (Σ_0^-, Σ_0^+) is not smooth, and not uniformely curved. The purpose of this subsection is to prove the following proposition:

Proposition 6.28. *Arbitrarily close to Σ_0^- (resp. Σ_0^+), there exists a topological Cauchy surfaces Σ_1^- (resp. Σ_1^+), which is convex (resp. concave) and uniformly curved.*

The idea of the proof of Proposition 6.28 is to replace the convex set $C_0 = C(\partial S_0)$ by its "Lorentzian ε-neighbourhood". This idea comes from Riemannian geometry. Indeed, it is well-known that the ε-neighbourhood of a convex subset of the hyperbolic space $\mathbb{H}^n$ is uniformly convex. We will prove that a similar phenomenon occurs in $\mathbb{A}d\mathbb{S}_3$ (although technical problems appear).

The *length* of a C^1 causal curve $\gamma : [0,1] \to \mathbb{A}d\mathbb{S}_3$ is $l(\gamma) = \int_0^1 (-g(\dot\gamma(t), \dot\gamma(t)))^{1/2} dt$, where g is the Lorentzian metric of $\mathbb{A}d\mathbb{S}_3$. Given an achronal subset E of $\mathcal{A}_{p_0}$ and a point p in $\mathcal{A}_{p_0}$, the *distance from p to E* is the supremum of the lengths of all the C^1 causal curves joining p to E in $\mathcal{A}_{p_0}$ (if there is no such curve, then the distance from p to E is not defined)[11] The distance from p to E, when finite, is lower semi-continuous in p. Moreover, the distance from p to E is continuous in p, when p is in the Cauchy development of E (see, for instance, [14, page 215]).

Given an achronal subset E of $\mathcal{A}_{p_0}$ and $\varepsilon > 0$, the *ε-future* of E is the set made of the points $p \in \mathcal{A}_{p_0}$, such that p is in the future of E and such that the distance from p to E is at most ε. We define similarly the *ε-past* of E. We denote by $I_\varepsilon^-(E)$ and $I_\varepsilon^+(E)$ the ε-past and the ε-future of the set E.

Lemma 6.29. *There exists $\varepsilon > 0$ such that the ε-past and the ε-future of the surface S_0^+ are contained in $D(S_0)$.*

Proof. Since the set $D(S_0)$ is a neighbourhood of the surface S_0^+, and since the surface $\Sigma_0^+ = \Gamma\backslash S_0^+$ is compact, one can find a Γ-invariant neighbourhood U_0^- of the surface S_0^+, such that U_0^+ is contained in $D(S_0)$, and such that $\Gamma\backslash U_0^+$ is compact.

Claim. There exists $\varepsilon > 0$ such that the distance from any point $p \notin U_0^+$ to the surface S_0^+ is bigger than ε.

By contradiction, suppose that, for every $n \in \mathbb{N}$, there exists a point $x_n \in \mathcal{A}_{p_0} \setminus U_0^+$ such that the distance from x_n to the surface S_0^+ is less than $1/n$. Then, for each n, we consider a causal curve γ_n joining the point x_n to the surface S_0^+. This curve γ_n must intersect the boundary of U_0^+; let z_n be a point in $\gamma_n \cap \partial U_0^+$. Since z_n is on a causal curve joining x_n to the surface S_0^+, the distance from z_n to S_0^+ must be smaller than $1/n$. Now, recall that $\Gamma\backslash U_0^+$ is compact. Hence, up to replacing each z_n by its image under some element of Γ, we may assume that all the z_n's are in a compact subset of the boundary of U_0^+. Then, we consider a limit point z of the sequence $(z_n)_{n\in\mathbb{N}}$. By lower semi-continuity of the distance, the distance from z to the surface

[11] The same definition work in the case where the set E is not achronal. But then, the distance from p to E might be positive even if $p \in E$!

S_0^+ is equal to zero (note that the distance from z to the surface S_0^+ is well-defined, since every point of $\mathcal{A}_{p_0}$ can be joined from the surface S_0^+ by a timelike curve, see Remark 5.13). Hence, the point z is on the surface S_0^+. This is absurd, since z must be on the boundary of U_0^+, and since U_0^+ is a neighbourhood of S_0^+. This completes the proof of the claim. The lemma follows immediately. □

Definition of the set C_1. From now on, we fix a number $\varepsilon > 0$ such that the ε-pasts and ε-futures of the surfaces S_0^- and S_0^+ are contained in $D(S_0)$. We consider the set

$$C_1 := C_0 \cup I_\varepsilon^-(S_0^-) \cup I_\varepsilon^+(S_0^+)$$

Obviously, C_1 is a Γ-neighbourhood of C_0 contained in $D(S_0)$. Actually, C_1 should be thought as a "Lorentzian ε-neighbourhood" of C_0.

Our aim is to prove that the boundary of the set $\Gamma\backslash C_1$ is made of two topological Cauchy surfaces which are convex/concave and uniformly curved. For that purpose, we first need to prove that C_1 is a convex set. Let us introduce some notations. We denote by $\mathcal{P}(S_0^-)$ (resp. by $\mathcal{P}(S_0^+)$) the set of the support planes of the surface S_0^- (resp. the surface S_0^+).

Lemma 6.30. *The set C_1 is made of the points $p \in \mathcal{A}_{p_0}$ such that:*
- for every plane P in $\mathcal{P}(S_0^+)$, the point p is in the past or in the ε-future of P,
- for every plane P in $\mathcal{P}(S_0^-)$, the point p is in the future or in the ε-past of P.
In other words:

$$C_1 = \left(\bigcap_{P \in \mathcal{P}(S_0^-)} I_\varepsilon^-(P) \cup I^+(P) \right) \cap \left(\bigcap_{P \in \mathcal{P}(S_0^+)} I^-(P) \cup I_\varepsilon^+(P) \right) \tag{1}$$

Proof. We denote by C_1' the right-hand term of equality (1). Let p be a point of $\mathcal{A}_{p_0}$ which is not in C_1'. Assume for instance that there exists a plane $P \in \mathcal{P}(S_0^+)$, such that p is in the future of P, and the distance from p to P is bigger than ε. Since the surface S_0^+ is in the past of P, this implies that p is in the future of S_0^+ and that the distance from p to Σ_0^+ is bigger than ε. Hence, p is not in C_1.

Conversely, let p be a point of $\mathcal{A}_{p_0}$ which is not in C_1. Assume for instance that p is in the future of the surface S_0^+ and the distance from p to S_0^+ is bigger than ε. Then there exists a timelike curve γ joining p to a point $q \in \Sigma_0^+$, such that the length of γ is bigger than ε. Let P be a support of C_0 such that $q \in P \cap C_0$. By definition, P is an element of $\mathcal{P}(S_0^+)$, the point p is in the future of P, and the distance from p to P is bigger than the length of γ. Hence, p is not in C_1'. □

Using the diffeomorphism Φ_{p_0} (see subsection 4.3), we identify the domain $\mathcal{A}_{p_0}$ with the region of $\mathbb{R}^3$ where $x^2 + y^2 - z^2 < 1$. Let P_0 be the totally geodesic subspace of $\mathcal{A}_{p_0}$ defined as the intersection of $\mathcal{A}_{p_0}$ with the affine plane $(z = 0)$ in $\mathbb{R}^3$. Obviously, P_0 is spacelike.

Lemma 6.31. *The set $I^-(P_0) \cup I_\varepsilon^+(P_0)$ is the region of $\mathcal{A}_{p_0}$ defined by the inequation*

$$z \leq \tan\varepsilon . \sqrt{1 - x^2 - y^2}$$

Proof. All the calculations have to be made in the linear model of the anti-de Sitter space, using the coordinates x_1, x_2, x_3, x_4 (because in this model the Lorentzian metric is simply the restriction of a global quadratic form). The equation of P_0 in this system of coordinates is $(x_1 = 0)$. The equation $(z = \tan\varepsilon . \sqrt{1 - x^2 - y^2})$ corresponds to the equation $(x_1 = \sin\varepsilon)$. On the one hand, since P_0 is a smooth spacelike surface, the distance from a point $q \in D(P_0)$ to the plane P_0 is realized as the length of a geodesic segment joining q to P_0 and orthogonal to P_0 (see, for instance, [14]). On the other hand, Proposition 4.2 implies every point q on the surface $(x_1 = \sin\varepsilon)$ belongs to a unique geodesic which is orthogonal to P_0. So, we are left to prove that, for every point p on P_0, the length of the unique segment of geodesic orthogonal to P_0 and joining p to the surface $(x_1 = \sin\varepsilon)$ is equal to ε. This follows from Proposition 4.2 and from an elementary calculation. $\square$

Remark 6.32. *Lemma 6.31 shows that $I^-(P_0) \cup I^+_\varepsilon(P_0)$ is a relatively convex subset of $\mathbb{A}d\mathbb{S}_3$ (that is, the intersection of a convex subset of $\mathbb{S}^3$ with $\mathbb{A}d\mathbb{S}_3$). Moreover, it shows that there exists R such that the boundary of the set $I^-(P_0) \cup I^+_\varepsilon(P_0)$ is more curved than a sphere of radius R at every point: if we consider the Euclidean metric on $\mathbb{R}^3$ for which (x, y, z) is an orthonormal system of coordinates, then we can take $R = (\tan\varepsilon)^{-1}$. Although this does not clearly appear in the proof of Lemma 6.31, this phenomenon is related with the negativity of the curvature of $\mathbb{A}d\mathbb{S}_3$.*

Corollary 6.33. *The set C_1 is convex.*

Proof. Consider a totally geodesic subspace $P \in \mathcal{P}^+(S_0^+)$. There exists $\sigma_P \in O_0(2,2)$, such that $\gamma_P(P_0) = P$. Then, σ_P maps the set $I^-(P_0) \cup I^+_\varepsilon(P_0)$ to the set $I^-(P) \cup I^+_\varepsilon(P)$. By Remark 6.32, the set $I^-(P_0) \cup I^+_\varepsilon(P_0)$ is relatively convex. Hence, the set $I^-(P) \cup I^+_\varepsilon(P)$ is also relatively convex. The same arguments show that, for every $P \in \mathcal{P}^-(C_0)$, the set $I^-_\varepsilon(P) \cup I^+(P)$ is relatively convex. Together with Lemma 6.30, this shows that the set C_1 is a relatively convex subset of $\mathbb{A}d\mathbb{S}_3$. Moreover, C_1 is contained in $D(S_0)$, which is a convex subset of $\mathbb{A}d\mathbb{S}_3$ (see item (iii) of Remark 5.16 and Proposition 6.16). Therefore, C_1 is a convex subset of $\mathbb{A}d\mathbb{S}_3$. $\square$

Definition of the surfaces S_1^-, S_1^+, Σ_1^- and Σ_1^+. The set C_1 is a Γ-invariant closed convex subset of $\mathbb{A}d\mathbb{S}_3$, containing C_0, and contained in $D(S_0)$. By Proposition 6.26, the boundary of C_1 in $\mathbb{A}d\mathbb{S}_3$ is the union of two Γ-invariant spacelike topological surfaces S_1^- and S_1^+, such that (S_1^-, S_1^+) is a convex trap. Also by Proposition 6.26, $(\Sigma_1^- := \Gamma\backslash S_1^-, \Sigma_1^+ := \Gamma\backslash S_1^+)$ is a convex trap.

Remark 6.34. *The surface S_1^- (resp. S_1^+) is the set made of all the points of $\mathcal{A}_{p_0}$ which are in the past of the surface S_0^- (resp. S_0^+), at distance exactly ε of S_0^- (resp. S_0^+): this follows from the definition of the set C_1, and from the continuity of the distance from a point p to the surface S_0^- (resp. S_0^+) when p ranges in $D(S_0) = D(S_0^-) = D(S_0^+)$. Thus, the surface Σ_1^- (resp. Σ_1^+) is the set made of all the points of M which are in the past of the surface Σ_0^- (resp. Σ_0^+), at distance exactly ε of Σ_0^- (resp. Σ_0^+).*

Proposition 6.35. *The surfaces Σ_1^- and Σ_1^+ are uniformly curved.*

Proof. Fix a Euclidean metric on $\mathbb{R}^3$, and let $\Delta_1^+ \subset S_1^+$ be a compact fundamental domain of the action of Γ on S_1^+. Let Δ_0^+ be the intersection of the past of Δ_1^+ with the surface S_0^+. Note that Δ_0^+ is compact (since Δ_1^+ is compact, and since Δ_1^+ and S_0^+ are contained in a globally hyperbolic subset of $\mathbb{A}d\mathbb{S}_3$). Let $\mathcal{P}(\Delta_0^+)$ be the set of all the support planes of S_0^+ that meet S_0^+ at some point of Δ_0^+.

Claim 1. There exists R such that, for every $P \in \mathcal{P}(\Delta_0^+)$, the boundary of the set $I^-(P) \cup I^+_\varepsilon(P)$ is more curved than a sphere of radius R.

On the one hand, $\mathcal{P}(\Delta_0^+)$ is a compact subset of the set of all spacelike totally geodesic subspaces of $\mathbb{A}d\mathbb{S}_3$. As a consequence, there exists a compact subset $\mathcal{K}$ of $O_0(2,2)$ such that $\mathcal{P}(\Delta_0^+) \subset \mathcal{K}.P_0$. On the other hand, there exists R_0 such that the boundary of the set $I^-(P_0) \cup I^+_\varepsilon(P_0)$ is more curved than a sphere of radius R_0 (see Remark 6.32). The claim follows.

Claim 2. Every $q \in \Delta_1^+$ is on the boundary of the set $I^-(P) \cup I^+_\varepsilon(P)$ for some P in $\mathcal{P}(\Delta_0^+)$.

Let $q \in \Delta_1^+ \subset S_1^+$. By definition of S_1^+, the point q is in the future of the surface S_0^+ and the distance from q to S_0^+ is ε. Moreover, since q and S_0^+ are contained in a globally hyperbolic region of $\mathbb{A}d\mathbb{S}_3$, the distance between q and S_0^+ is realized: there exists a causal curve γ of length ε joining q to a point $p \in S_0^+$. By construction, the point p is in Δ_0^+. Let P be any support plane of S_0^+ at p. Of course, P is in $\mathcal{P}(\Delta_0^+)$. On the one hand, since γ is a causal arc of length ε joining q to a point of P, the distance from p to P is at least ε. On the other hand, Lemma 6.30 implies that the distance from p to P must be at most ε. The claim follows.

Let q be a point of Δ_1^+. By claim 2, there exists $P \in \mathcal{P}(\Delta_0^+)$ such that q is on the boundary of the set $I^-(P) \cup I_\varepsilon^+(P)$. By Lemma 6.30, the surface S_1^+ is contained in $I^-(P) \cup I_\varepsilon^+(P)$. Putting these together with claim 1, we obtain that the surface S_1^+ is more curved than a sphere of radius R at q. Hence, the surface Σ_1^+ is uniformly curved. □

This completes the proof of Proposition 6.28.

Remark 6.36. *All the results of this subsection are still valid if one replaces (Σ_0^-, Σ_0^+) by any other convex trap.*

Remark 6.37. *It is well-known that the boundary of the ε-neighbourhood of any geodesically convex subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ is a C^1 hypersurface. Unfortunately, this phenomenon does not generalize to Lorentzian setting. In particular, the surfaces S_1^-, S_1^+, Σ_1^- and Σ_1^+ are not C^1 in general.*

6.6 Smoothing the Cauchy surfaces Σ_1^- and Σ_1^+

In order to apply Gerhard's theorem, we need a *smooth* uniformly curved convex trap. The purpose of this subsection is to prove the following proposition:

Proposition 6.38. *Arbitrarily close to Σ_1^- and Σ_1^+, there exist some C^∞ Cauchy surfaces Σ^- and Σ^+, such that Σ^- is convex and uniformly curved and Σ^+ is concave and uniformly curved.*

Unfortunately, we could not find any simple proof of Proposition 6.38 (see Remark 6.39). Our proof is divided in three steps. In 6.6.1, we approximate the surfaces Σ_1^- and Σ_1^+ by some *polyhedral* Cauchy surfaces Σ_2^- and Σ_2^+ (respectively convex and concave). Then, in 6.6.2, we describe a method for smoothing convex and concave polyhedral Cauchy surfaces. Using this method, we obtain two disjoint C^∞ Cauchy surfaces Σ_3^- and Σ_3^+, respectively convex and concave. Finally, in 6.6.3, using the same trick as in subsection 6.5, we get a smooth uniformly convex trap.

Remark 6.39. *The first idea which comes to in mind for smoothing a convex surface is to use some convolution process. Unfortunately, to make this kind of idea work, one needs a locally Euclidean structure*[12]*. This is the reason why this kind of idea does not fit our situation (there is no locally Euclidean structure on the manifold M).*

6.6.1 Polyhedral convex and concave Cauchy surfaces

In this subsubsection, we will define a notion of *polyhedral surface* in M. Then, we will construct two polyhedral Cauchy surfaces Σ_2^- and Σ_2^+ in M, such that Σ_2^- is convex, Σ_2^+ is concave, and Σ_2^+ is in the future of Σ_2^-.

A subset Δ of M is a 2-*simplex*, if there exists a projective chart $\Phi : U \subset M \to \mathbb{R}^3$, such that $\Delta \subset U$ and such that $\Phi(\Delta)$ is a 2-simplex in $\mathbb{R}^3$. A compact surface Σ in M is called *polyhedral* if it can be decomposed as a finite union of 2-simplices.

[12] For example, any convex function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated by a smooth convex function $\widehat{f}$, obtained as the convolution of f with an approximation of the unity, but the proof of the convexity of $\widehat{f}$ uses the Euclidean structure of $\mathbb{R}^{n+1}$.

Remark 6.40. *Let Σ be a compact spacelike surface in M, let $\widetilde{\Sigma}$ be a lift of Σ in $\widetilde{M}$, and let $S := \mathcal{D}(\widetilde{\Sigma})$. Using the embedding $\Phi_{p_0} : \mathcal{A}_{p_0} \to \mathbb{R}^3$, we can see S as a surface in $\mathbb{R}^3$. Then, Σ is a polyhedral surface if and only if S can be decomposed as a finite union of orbits (for Γ) of 2-simplices of $\mathbb{R}^3$.*

Remark 6.41. *Let Σ be a compact convex spacelike polyhedral surface in M. Then, Σ can be decomposed as a finite union of subsets $\Sigma := \Delta_1 \cup \cdots \cup \Delta_n$, where each Δ_i is the intersection of Σ with one of its support planes, and each Δ_i has non-empty interior (as a subset of Σ). The decomposition is unique (provided that the Δ_i's are pairwise distinct). The Δ_i's are called the* sides *of Σ. Each side of Σ is a finite union of 2-simplices, but is not necessarily a topological disc (e.g. in the case where Σ is totally geodesic).*

Definition of the set C_2, of the surfaces S_2^-, S_2^+, Σ_2^- and Σ_2^+ We consider a Γ-invariant set E of points of $\partial C_1 = S_1^- \cup S_1^+$, such that $\Gamma \backslash E$ is finite (in particular, E is discrete). We denote by C_2 the convex hull of E. By construction, C_2 is a Γ-invariant convex subset of C_1. In particular, C_2 is a Γ-invariant convex subset of $D(S_0)$. So, by Proposition 6.26, the pair of boundary components of C_2 in $\mathbb{A}d\mathbb{S}_3$, and their projections in M, are convex traps.

Given $\delta > 0$, we say that the set E is *δ-dense in the surfaces S_1^- and S_1^+*, if every Euclidean ball of radius δ centered at some point of S_1^- (resp. S_1^+) contains some points of E. The remainder of the subsection is devoted to the proof of the following proposition:

Proposition 6.42. *There exists $\delta > 0$ such that, if the set E is δ-dense in the surfaces S_1^- and S_1^+, then the surfaces Σ_2^- and Σ_2^+ are polyhedral.*

Remark 6.43. *The proof of Proposition 6.42 is quite technical. The reader who is not interested in technical details can skip the proof. Nevertheless, it should be noticed that the boundary of the convex hull of a discrete set of points is* not *a polyhedral surface in general. In particular, Proposition 6.42 would be false if the surfaces Σ_1^- and Σ_1^+ were not uniformly curved.*

Given a set $F \subset \mathbb{R}^3$, we say that an affine plane P of $\mathbb{R}^3$ *splits* the set F, if F intersects the two connected components of $\mathbb{R}^3 \setminus P$. The starting point of the proof of Proposition 6.42 is the following well-known fact (which follows from very basic arguments of affine geometry):

Fact 6.44. *For every finite set of points $F \subset \mathbb{R}^3$, the boundary of $Conv(F)$ is a compact polyhedral surface; more precisely, the boundary of $Conv(F)$ is the union of all the 2-simplices $Conv(p, q, r)$, such that the points p, q, r are in F, and such that the plane (p, q, r) does not split F.*

Let γ be a continuous curve in a Euclidean plane, and p be a point on γ. We say that the curve γ is *more curved than a circle of radius R at p* if there exists a Euclidean disc Δ of radius R, such that p is on the boundary of Δ, and such that Δ contains a neighbourhood of p in γ. The proof of the following lemma of elementary planar geometry is left to the reader:

Lemma 6.45. *Given two positive numbers ρ and R, there exists a positive number $\delta = \delta(\rho, R)$ such that: for every convex set D in an Euclidean plane, if there exists a subarc α of the boundary of D, such that the boundary of D is more curved than a circle of radius R at each point of α, and such that the diameter of α is bigger than ρ, then D contains a Euclidean ball of radius δ.*

Proof of Proposition 6.42. Consider a compact fundamental domain U for the action of Γ on C_1. Then, consider a compact neighbourhood V of U in C_1, and a compact neighbourhood W of V in C_1. One can find a positive number ρ such every Euclidean ball of radius ρ centered in U (resp. V) is contained in V (resp. W). Moreover, since V is compact, one can find a positive number R, such that the surface S_1^- (resp. S_1^+) is more curved than a sphere of radius R at every point of $S_1^- \cap V$ (resp. $S_1^+ \cap V$).

From now on, we assume that the set E is δ-dense in the surfaces S_1^- and S_1^+, where $\delta = \delta(\rho, R)$ is the positive number given by Lemma 6.45. Up to replacing δ by $\min(\delta, \rho)$, we can assume that δ is smaller than ρ. Under these assumptions, we shall prove that the surfaces S_2^- and S_2^+ are polyhedral.

Claim 1. If p, q, r are three points of E, such that the 2-simplex Conv(p, q, r) intersects U, and such that the affine plane $P := (p, q, r)$ does not split the set E, then the three points p, q, r are in W.

To prove this claim, we argue by contradiction: we suppose that there exists three points p, q, r in E, such that the 2-simplex $\mathrm{Conv}(p, q, r)$ intersects U, such that the affine plane $P := (p, q, r)$ does not split the set E, and such that one of the three points p, q, r is not in W. We shall show that these assumptions contradict the δ-density of the set E.

Since P does not split the set E, one of the two connected components of $\mathcal{A}_{p_0} \setminus P$ is disjoint from E. We denote by H_P this connected component. First of all, we observe that H_P does not intersect the curve ∂S_0, since H_P does not contain any point of E, since E is a non-empty Γ-invariant subset of $D(S_0)$, and since the curve ∂S_0 is the limit set of the action of Γ on $D(S_0)$. Therefore, the intersection of H_P with the boundary of C_1 is contained in one of the two connected components S_1^- and S_1^+ of $\partial C_1 \setminus \partial S_0$. Without loss of generality, we assume that $H_P \cap \partial C_1$ is contained in S_1^+, and we consider the set $D^+ := H_P \cap S_1^+$ (see figure 2).

We shall prove that there exists an Euclidean ball B of radius δ centered at some point of D^+, such that $B \cap S_1^+ \subset D^+$. Since D^+ must be disjoint from E (because $D^+ \subset H_P$), this will contradict the fact that E is δ-dense in S_1^+. For that purpose, we consider the curve $\gamma := P \cap S_1^+$. Observe that this curve γ is the boundary of the topological disc D^+. Moreover, the curve γ is also the boundary of the convex subset $D := P \cap C_1$ of the plane P. The curve γ passes through the points p, q and r, and the 2-simplex $\mathrm{Conv}(p, q, r)$ is contained in the convex set D. We shall distinguish two cases (and get a contradiction in each case):

First case: the curve γ does not intersect the neighbourhood V. We consider a point m in $D \cap U$ (such a point does exist since $\mathrm{Conv}(p, q, r) \cap U \neq \emptyset$ and $\mathrm{Conv}(p, q, r) \subset D$), and we denote by m' the unique point of intersection of D^+ with the line passing through m and orthogonal to the plane P. The point m is in U, and the curve γ does not intersect V; so, by definition of ρ, the Euclidean distance between m and γ must be bigger than ρ, and thus, bigger than δ. Moreover, the Euclidean distance between the point m' and the curve γ is bigger than the distance between m and γ. So, we have proved that the Euclidean ball B of radius δ centered at m' does not intersect the curve γ. Hence, the connected component of $B \cap S_1^+$ containing the point m' is contained in D^+. Since D^+ is disjoint from E, this contradicts the δ-density of E in S_1^+.

Second case: the curve γ does intersect the neighbourhood V. Then, by definition of ρ, we can find a subarc α of the curve γ, such that the diameter of α is bigger than ρ, and such that α is contained in W. Since S_1^+ is more curved than a sphere of radius R at every point of V, the curve γ is more curved than a circle of radius R at each point of α. Thus, by lemma 6.45, there exists a point $m \in D$ such that the Euclidean distance between the point m and the curve γ is bigger than δ. The same argument as above shows that this contradicts the δ-density of E in the surface S_1^+.

In both case, we have obtained a contradiction. This completes the proof of claim 1.

Claim 2. If W' is a compact subset of $\mathcal{A}_{p_0}$ such that $W \subset W'$, then the sets Conv$(E \cap W') \cap U$ and Conv$(E \cap W) \cap U$ coincide.

This claim is a consequence of Claim 1 and fact 6.44. Since W' is a compact subset of $\mathbb{A}d\mathbb{S}_3$, the set $E \cap W'$ is finite. Hence, the boundary of the set $\mathrm{Conv}(E \cap W')$ is the union of the 2-simplices

$[p, q, r]$, such that the three points p, q, r are in $E \cap W'$, and such that the affine plane (p, q, r) does not split $E \cap W'$. By claim 1, such a 2-simplex can intersect U only if the three points p, q and r are in W. Using once again fact 6.44, this implies that the intersection of boundary of $\mathrm{Conv}(E \cap W')$ with U is contained in the intersection of the boundary of $\mathrm{Conv}(E \cap W)$ with U. But, if the boundary of a convex set is contained in the boundary of another convex set, then these two convex sets must coincide. The claim follows.

End of the proof. Let us consider an increasing sequence $(W_n)_{n\in\mathbb{N}}$ of compacts subsets of $\mathbb{A}d\mathbb{S}_3$, such that $\bigcup_{n\in\mathbb{N}} W_n = \mathbb{A}d\mathbb{S}_3$. On the one hand, we clearly have $\mathrm{Conv}(E) = \mathrm{Closure}(\bigcup_{n\in\mathbb{N}} \mathrm{Conv}(E \cap W_n))$. On the other hand, according to Claim 2, there exists an integer n_0 such that $\mathrm{Conv}(E \cap W_n) \cap U = \mathrm{Conv}(E \cap W) \cap U$ for every $n \geq n_0$. As a consequence, we have $\mathrm{Conv}(E) \cap U = \mathrm{Conv}(E \cap W) \cap U$. Now, since $E \cap W$ is a finite set, the boundary of $\mathrm{Conv}(E \cap W)$ is a compact polyhedral surface. Thus, we have proved that the boundary of the set $C_2 = \mathrm{Conv}(E)$ coincides, in U, with a polyhedral surface. Since U contains a fundamental domain for the action of Γ on C_2, this implies each of the surfaces S_2^- and S_2^+ can be decomposed as a finite union of orbits of 2-simplices. Hence, the surfaces Σ_2^- and Σ_2^+ are polyhedral (see Remark 6.40). ☐

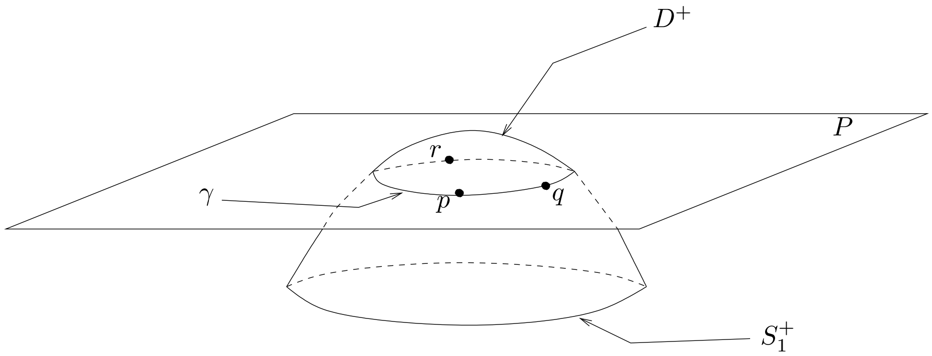

Figure 2: The situation in the proof of Proposition 6.42

Addendum. *There exists $\delta > 0$ such that, if the set E is δ-dense in the surfaces S_1^+, S_1^-, then each side of the polyhedral surfaces Σ_2^-, Σ_2^+ is contained in the domain of an projective chart of M.*

Proof. From the proof of Proposition 6.42, one can extract the following statement: for every $\rho > 0$, there exists $\delta > 0$ such that, if the set E is δ-dense in the surface S_1^-, then, for every support plane P of the surface S_2^-, the diameter of the set $P \cap S_2^-$ is less than ρ. Of course, there is a similar statement for the surface S_2^+. The addendum follows immediately. ☐

6.6.2 Smooth convex and concave Cauchy surfaces

In this subsubsection, we describe a process for smoothing the polyhedral Cauchy surfaces Σ_2^- and Σ_2^+. More precisely, we prove the following:

Proposition 6.46. *Let Σ be a convex polyhedral Cauchy surface in M. Assume that each side of Σ is contained in an affine domain of M. Then, arbitrarily close to Σ, there exists a C^∞ convex Cauchy surface.*

Of course, the analogous statement dealing with concave Cauchy surfaces is also true. The proof of Proposition 6.46 relies on the following technical lemma:

Lemma 6.47. *Let U be some subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a continuous convex function. Then, for every $\eta > 0$, there exists a continuous convex function $\widehat{f} : U \to \mathbb{R}$ satisfying the following properties:*

- *$\widehat{f} \geq f$, the distance between f and $\widehat{f}$ is less than 2η, and $\widehat{f}$ coincides with f on the set $f^{-1}([2\eta, +\infty[)$;*
- *$\widehat{f}$ is constant on the set $f^{-1}([0, \eta])$; in particular, $\widehat{f}$ is C^∞ on the set $f^{-1}([0, \eta[)$;*
- *if f is C^∞ on some subset U of Dom(f), then $\widehat{f}$ is also C^∞ on U.*

Proof. We consider a C^∞ function $\varphi : [0, +\infty[\to [0, +\infty[$ such that: φ is non-decreasing and convex, $\varphi(t) = \frac{3}{2}\eta$ for every $t \in [0, \eta]$, and $\varphi(t) = t$ for every $t \in [2\eta, +\infty[$. Then, consider $\widehat{f} : U \to [0, +\infty[$ defined by $\widehat{f} := \varphi \circ f$. This function satisfies all the required properties. □

We endow M with a Riemannian metric; this allows us to speak of the (Riemannian) ε-neighbourhood of any subset of M for any $\varepsilon > 0$. We say that a surface Σ_1 is *ε-close* to another surface Σ_2 if there exists a homeomorphism $\Psi : \Sigma_1 \to \Sigma_2$ which is ε-close to the identity. The following Remark will allow us to see a polyhedral surface as a collection of graphs of functions:

Remark 6.48. *Let Σ be a convex compact surface in M, let Π be a support plane of Σ and let $\Delta := \Sigma \cap \Pi$. We assume that Δ is contained in an affine domain of M. Then, we can find a neighbourhood V of F in M, and some local affine coordinates (x, y, z) on V, such that:*

– $\Pi \cap V$ is the plane of equation $(z = 0)$, and $\Sigma \cap V$ is the graph $(z = f(x, y))$ of a non-negative convex function $f : U \to [0, +\infty[$ (where U is some convex subset of $\mathbb{R}^2$).

– if Σ' is a convex Cauchy surface close enough to Σ, then $\Sigma' \cap V$ is the graph $z = f'(x, y)$ of a convex function $f' : U \to \mathbb{R}$. The function f' depends continuously of the surface Σ'. Moreover, if Σ' is in the future of Σ, then $f' \geq f$ (and thus, $f' \geq 0$).

We denote by $\Delta_1, \ldots, \Delta_n$ the sides of the polyhedral surface Σ. To prove Proposition 6.46, we will construct a sequence of convex Cauchy surfaces $\Sigma_0, \ldots, \Sigma_n$, where $\Sigma_0 = \Sigma$, and where Σ_{k+1} is obtained by smoothing Σ_k in the neighbourhood of Δ_{k+1}. More precisely, we will prove the following:

Proposition 6.49. *For every $k \in \{0, \ldots, n\}$ and every $\varepsilon > 0$ small enough, there exists a convex Cauchy surface $\Sigma_{k,\varepsilon}$ in M such that:*

– the surface $\Sigma_{k,\varepsilon}$ is in the future of the surface Σ.

– the surface $\Sigma_{k,\varepsilon}$ is ε-close to the surface Σ,

– the surface $\Sigma_{k,\varepsilon}$ is smooth outside the ε-neighbourhoods of the sides $\Delta_{k+1}, \ldots, \Delta_n$.

Notice that Proposition 6.49 implies Proposition 6.46 (for $k = n$, the surface $\Sigma_{k,\varepsilon}$ is a smooth convex Cauchy surface, ε-close to the initial surface Σ). So, we are left to prove Proposition 6.49.

Proof of Proposition 6.49. First of all, we set $\Sigma_{0,\varepsilon} := \Sigma$ for every $\varepsilon > 0$. Now, let $k \in \{0, \ldots, n-1\}$, and let us suppose that we have constructed the surface $\Sigma_{k,\varepsilon}$ for every $\varepsilon > 0$ small enough. We will construct the surface $\Sigma_{k+1,\varepsilon}$ for every $\varepsilon > 0$ small enough.

Since Δ_{k+1} is a side of Σ, there exists a support plane Π_{k+1} of Σ such that $\Pi_{k+1} \cap \Sigma = \Delta_{k+1}$. Using Remark 6.48, we find a compact neighbourhood V of Δ_{k+1} in M, and some local affine coordinates (x, y, z) on V, such that in these coordinates, $\Pi_{k+1} \cap V$ is the plane of equation $(z = 0)$, and the surface $\Sigma \cap V$ is the graph $(z = f(x, y))$ of a non-negative convex function $f : \mathrm{Dom}(f) \subset \mathbb{R}^2 \to \mathbb{R}$. Moreover, the function f is positive in restriction to $\partial\mathrm{Dom}(f)$, and thus, the quantity $\delta := \inf\{f(x, y) \mid (x, y) \in \partial\mathrm{Dom}(f)\}$ is positive ($\partial\mathrm{Dom}(f)$ is compact).

Now, we fix some $\varepsilon > 0$ such that $\varepsilon/3 < \delta/2$. By the second item of Remark 6.48, we can find $\varepsilon' > 0$, such that $\varepsilon' < \varepsilon/3$, and such that the surface $\Sigma_{k,\varepsilon'} \cap V$ is the graph of a convex function

$g : \mathrm{Dom}(g) = \mathrm{Dom}(f) \to \mathbb{R}$. Moreover, since $\Sigma_{k,\varepsilon'}$ is in the future of Σ, the function g is bigger than f; in particular, g is non-negative, and we have $g(x,y) > \delta$ for every $(x,y) \in \partial\mathrm{Dom}(g)$.

Applying Lemma 6.47 to the function g with $\eta := \varepsilon/3$, we obtain a convex function $\hat{g} : \mathrm{Dom}(g) \to [0,+\infty[$ satisfying the following properties:

(a) $\widehat{g} \geq g$ and the distance between g and $\widehat{g}$ is less than $2\varepsilon/3$,

(b) $\widehat{g}$ is C^∞ on $g^{-1}([0,\varepsilon/3])$,

(c) if g is smooth on some open subset of $\mathrm{Dom}(g) = \mathrm{Dom}(\widehat{g})$, then $\widehat{g}$ is also smooth on U,

(d) $\widehat{g}$ coincides with g on $g^{-1}([2\varepsilon/3,+\infty[)$; in particular, $\widehat{g}$ coincide with g on $\partial\mathrm{Dom}(\widehat{g}) = \partial\mathrm{Dom}(g)$.

We construct the surface $\Sigma_{k+1,\varepsilon}$ as follows: starting from the surface $\Sigma_{k,\varepsilon'}$, we cut off $\Sigma_{k,\varepsilon} \cap V$ (*i.e.* we cut off the graph of g), and we paste the graph of $\widehat{g}$. This is possible since the graphs of the functions g and $\widehat{g}$ coincide near the boundary of V (property (d)). There is a natural diffeomorphism Ψ between the surfaces $\Sigma_{k,\varepsilon'}$ and $\Sigma_{k+1,\varepsilon}$ defined as follows: Ψ coincides with the identity outside V, and maps the point of coordinates $(x,y,g(x,y))$ to the point of coordinates $(x,y,\widehat{g}(x,y))$. By property (a), Ψ is $(2\varepsilon/3)$-close to the identity; hence, the surface $\Sigma_{k+1,\varepsilon}$ is $(2\varepsilon/3)$-close to the surface $\Sigma_{k,\varepsilon'}$. Since $\Sigma_{k,\varepsilon'}$ is ε'-close to Σ, and since $\varepsilon' < \varepsilon/3$, we get that $\Sigma_{k+1,\varepsilon}$ is ε-close to Σ.

The inequality $\widehat{g} \geq g$ implies that $\Sigma_{k+1,\varepsilon}$ is in the future of $\Sigma_{k,\varepsilon'}$, and *a fortiori* in the future of Σ. The convexity of the function $\widehat{g}$ implies that $\Sigma_{k+1,\varepsilon}$ admits a support plane at each of its points. By Proposition 6.25 and Remark 6.20, this implies that $\Sigma_{k+1,\varepsilon}$ is a spacelike surface. Hence, $\Sigma_{k+1,\varepsilon}$ is a Cauchy surface (every compact spacelike surface embedded in M is a Cauchy surface). Now, since $\Sigma_{k+1,\varepsilon}$ is a spacelike surface admitting a support plane at each point, it is either convex or concave; and since it coincides with $\Sigma_{k,\varepsilon}$ outside V, it cannot be concave. So, $\Sigma_{k+1,\varepsilon}$ is a convex Cauchy surface.

It remains to study the smoothness of $\Sigma_{k+1,\varepsilon}$. Let q be a point on the surface $\Sigma_{k+1,\varepsilon}$, which is not in the union of the ε-neighbourhoods of the sides $\Delta_{k+2},\dots,\Delta_n$, and let $p := \Psi^{-1}(q) \in \Sigma_{k,\varepsilon'}$. Since the distance between the points p and q is less than $2\varepsilon/3$, the point p cannot be in the union of the $\varepsilon/3$-neighbourhoods of the sides $\Delta_{k+2},\dots,\Delta_n$. There are two cases:

– *if the point p is in the $\varepsilon/3$-neighbourhood of the side* Δ_{k+1}, then the distance between p and the plane Π_{k+1} is less than $\varepsilon/3$, and thus, property (b) implies that the surface $\Sigma_{k+1,\varepsilon}$ is smooth in the neighbourhood of $\Psi(p) = q$;

– *if the point p is not in the $\varepsilon/3$-neighbourhood of the side* Δ_{k+1}, then the surface $\Sigma_{k,\varepsilon'}$ is smooth in the neighbourhood of p (here, we use the inequality $\varepsilon' < \varepsilon/3$); hence, property (c) implies that the surface $\widehat{\Sigma}_{k,\varepsilon}$ is smooth in the neighbourhood of $\Psi(p) = q$.

As a consequence, the surface $\Sigma_{k+1,\varepsilon}$ is smooth outside the union of the ε-neighbourhoods of the sides $\Delta_{k+2},\dots,\Delta_n$. Therefore, the surface $\Sigma_{k+1,\varepsilon}$ satisfies all the required by properties. $\square$

Applying Proposition 6.46 to the polyhedral Cauchy surfaces Σ_2^- and Σ_2^+, we get two disjoint C^∞ Cauchy surfaces Σ_3^- and Σ_3^+, respectively convex and concave, such that Σ_3^+ is in the future of Σ_3^-.

6.6.3 Smooth uniformly curved convex and concave Cauchy surfaces

The Cauchy surfaces Σ_3^- and Σ_3^+ are smooth, respectively convex and concave, but not uniformly curved. Using the same trick as in subsection 6.5, we will aproximate Σ_3^- and Σ_3^+ by some smooth uniformly curved Cauchy surfaces Σ_4^- and Σ_4^+.

Definition of the Cauchy surfaces Σ_4^- and Σ_4^+. Let ε be a positive number. Let Σ_4^+ be the set made of the points $p \in M$, such that p is in the past of the surface Σ_3^+ and such that the distance from p to Σ_3^+ is exactly ε. If ε is small enough, then Σ_4^+ is a topological Cauchy surface which is convex and uniformly curved (see Remark 6.36 and Remark 6.34). We construct similarly a topological Cauchy surface Σ_4^- which is concave, uniformly curved, and contained in the past of Σ_3^-. By construction, Σ_4^+ is in the future of Σ_4^-.

Proposition 6.50. *If ε is small enough, the Cauchy surfaces Σ_4^- and Σ_4^+ are smooth (of class C^∞).*

Proof. We denote by TM the tangent bundle of M, by π the canonical projection of TM on M, and by $(\varphi^t)_{t\in\mathbb{R}}$ the geodesic flow on TM. We consider the subset $T_N\Sigma_3^+$ of TM made of the pairs (p,ν) such that p is a point of the surface Σ_3^+ and ν is the future-pointing unit normal vector of Σ_3^+ at p.

Let p be a point on the surface Σ_4^+. By construction of Σ_4^+, the distance from p to Σ_3^+ is exactly ε. Since M is globally hyperbolic, and since Σ_3^+ is a smooth spacelike surface, this implies that there exists a timelike geodesic segment of length exactly ε, orthogonal to Σ_3^+, joining Σ_3^+ to p (see, for example, [14, page 217]). As a consequence, the surface Σ_4^+ is contained in the set $\pi(\varphi^\varepsilon(T_N\Sigma_3^+))$.

We are left to prove that the set $\pi(\varphi^\varepsilon(T_N\Sigma_3^+))$ is a smooth surface. Since Σ_3^+ is a smooth compact spacelike surface in M, $T_N\Sigma_3^+$ is a smooth compact surface in TM, nowhere tangent to the fibers of the projection π, and hence, for ε small enough, $\varphi^\varepsilon(T_N\Sigma_3^+)$ is a smooth compact surface in TM, nowhere tangent to the fibers of π. Therefore, $\pi(\varphi^\varepsilon(T_N\Sigma_3^+))$ is a smooth surface in M. □

6.7 End of the proof of Theorem 1.1 in the case $g \geq 2$

In the previous paragraph, we have constructed a smooth uniformly curved convex trap (Σ_4^+, Σ_4^+). By Proposition 6.24, the surface Σ_4^- have negative curvature and the surface Σ_4^+ have positive curvature. Thus, (Σ_4^-, Σ_4^+) is a pair of barriers in M. By Theorem 6.1 and 6.3, the existence of a pair of barriers implies the existence of a CMC time function. This completes the proof of Theorem 1.1 in the case where the genus of the Cauchy surfaces is at least 2.

7 Proof of Theorem 1.1 in the case $g = 1$

The purpose of this section is to prove Theorem 1.1 in the case where the genus of the Cauchy surfaces of the spacetime under consideration is 1. According to Remark 2.4, after performing some finite covering if necessary, we can reduce this case to the case where the Cauchy surface is a 2-torus.

In subsection 7.1, we define a class of spacetimes, called Torus Universes[13], and we will prove that Torus Universes admit CMC time functions (actually, we construct explicitly a CMC time function on any such spacetime). Then, in subsection 7.2, we prove that every maximal globally hyperbolic spacetime, locally modelled on AdS_3, whose Cauchy surfaces are 2-tori, is isometric to a Torus Universe.

7.1 Torus Universes

Consider the 1-parameter subgroup of $SL(2,\mathbb{R})$ of diagonal matrices $(g^t)_{t\in\mathbb{R}}$ where:

$$g^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = e^{t\Delta} \quad \text{where:} \quad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[13]These spacetimes were already considered by several authors, see Remark 7.7

We denote by A the set of elements of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ for which both left and right components belong to the one-parameter subgroup $(g^t)_{t\in\mathbb{R}}$. Obviously, A is a free abelian Lie subgroup of rank 2 of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. This group acts isometrically on AdS_3 (recall that the isometry group of AdS_3 can be identified with $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$, see subsection 4.2). We denote by Ω the union of spacelike A-orbits in AdS_3.

We will see below that Ω has four connected components which are open convex domains of AdS_3. For any lattice $\Gamma\subset A$, the action of Γ on Ω is obviously free and properly discontinuous, and preserves each of the four connected components of Ω.

Definition 7.1. *A Torus Universe is the quotient $\Gamma\backslash U$ of a connected component U of Ω by a lattice Γ of A.*

Theorem 7.2. *Every Torus Universe is a globally hyperbolic spacetime, which admits a CMC time function.*

To prove Theorem 7.2, we will use the $SL(2,\mathbb{R})$-model of AdS_3 (see subsection 4.5). We recall that $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ acts on $SL(2,\mathbb{R})$ by $(g_L,g_R).g=g_Lgg_R^{-1}$.

Lemma 7.3. *For every element $g\in\Omega$, the A-orbit contains a unique element of the form*

$$R_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\qquad \textit{with}\qquad \theta\in[0,2\pi[$$

When g ranges over Ω, the angle θ varies continously with g, and ranges over $]0,\pi/2[\cup]\pi/2,\pi[\cup]\pi,3\pi/2[\cup]3\pi/2,2\pi[$.

Proof. Consider an element g in $AdS_3\simeq SL(2,\mathbb{R})$ and write

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\quad\text{with}\quad ad-bc=1$$

Then, the elements of the A-orbit of g are the matrices

$$g^tgg^{-s}=\begin{pmatrix}ae^{t-s} & be^{t+s}\\ ce^{-(t+s)} & de^{s-t}\end{pmatrix}$$

where s and t range over $\mathbb{R}$. Thus, the A-orbit of g is spacelike if and only if, for every $p,q\in\mathbb{R}$, the determinant of:

$$\begin{pmatrix}(p-q)a & (p+q)b\\ -(p+q)c & (q-p)d\end{pmatrix}$$

is negative, *i.e.* if and only if the quadratic form $(p-q)^2ad-(p+q)^2bc$ is positive definite. Since $ad-bc=1$, it follows that the A-orbit of g is spacelike if and only if:

$$\begin{array}{rcl}0< & ad & <1\\ -1< & bc & <0\end{array}$$

In particular, if the A-orbit of g is spacelike, then $abcd\neq 0$. It follows that, if the A-orbit of g is spacelike, then it contains an element of the form

$$R_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

(take s,t such that $e^{2(t-s)}=d/a$ and $e^{2(s+t)}=c/b$). The angle θ is obviously unique, it is not a multiple of $\frac{\pi}{2}$ (since $d\neq 0$ and $c\neq 0$), it varies continuously with g, and it takes any value in $[0,2\pi[$ that is not a multiple of $\frac{\pi}{2}$ when g ranges over Ω. $\square$

Remark 7.4. *If* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Omega$*, then the unique number* $\theta \in [0, 2\pi[$ *such that the rotation* R_θ *is in the* A*-orbit of* g *is characterized by the equalities* $\cos^2\theta = ad$ *and* $-\sin^2\theta = bc$ *(see the proof of Lemma 7.3).*

Lemma 7.3 implies that Ω has four connected components (corresponding to $\theta \in]0, \frac{\pi}{2}[$, $\theta \in]\frac{\pi}{2}, \pi[$, $\theta \in]\pi, \frac{3\pi}{2}[$, and $\theta \in]\frac{3\pi}{2}, 2\pi[$).

Remark 7.5. *The four connected components of* Ω *are all isometric one to the other by isometries centralizing the group* A*. Hence, with no loss of generality, we may restrict ourselves to Torus Universes that are obtained as quotients of the connected component corresponding to* $0 < \theta < \pi/2$*.*

Proof of Theorem 7.2. Denote by U the connected component of Ω corresponding to $0 < \theta < \frac{\pi}{2}$. Consider a lattice Γ in A, and consider the associated Torus Universe $M = \Gamma\backslash U$. Lemma 7.3 provides us with a continuous function $\theta : U \mapsto]0, \frac{\pi}{2}[$. By construction, this function is increasing with time and Γ-invariant: it follows that the quotient manifold $M = \Gamma\backslash U$ is equipped with a time function $\bar{\theta}$.

The equalities $\cos^2\theta = ad$ and $-\sin^2\theta = bc$ (see Lemma 7.3) imply that the connected component U is exactly

$$\left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \text{ such that } 0 < a\ ,\ 0 < b\ ,\ 0 > c \text{ and } 0 < d \right\}$$

Thus, in the Klein model of $\mathbb{A}d\mathbb{S}_3$, the connected component U is the interior of a simplex which is the convex hull of four points in $\partial\mathbb{A}d\mathbb{S}_3$ (these points are nothing but the fixed points of A) (see figure 3). The main information we extract from this observation is that U is a convex domain in $\mathbb{A}d\mathbb{S}_3$, in particular, its intersection with any geodesic - in particular, nonspacelike geodesics - is connected. Moreover, geodesics joining two points of ∂U satisfying both $bc = 0$ (respectively $ad = 0$) are spacelike. Hence, nonspacelike segments in U admits two extremities in ∂U, one satisfying $bc = 0$, and the other $ad = 0$. The equalities $ad = \cos^2\theta$, $bc = -\sin^2\theta$ imply that θ restricted to such a nonspacelike segment takes all values between 0, and $\frac{\pi}{2}$. In other words, every nonspacelike geodesic in U intersects every fiber of θ. Hence, every nonspacelike geodesic in M intersects every fiber of $\bar{\theta}$: these fibers are thus Cauchy surfaces, and M is globally hyperbolic.

Since every fiber of $\bar{\theta}$ is a A-orbit, it obviously admits constant mean curvature $\kappa(\bar{\theta})$. Let us calculate this mean curvature at R_θ. We will need to take covariant derivatives, and here, the situation is similar to the familiar situation concerning Riemannian embeddings: if X, Y are vector fields in $M(2, \mathbf{R})$ both tangent to G, then the covariant derivative $\overline{\nabla}_X Y$ in G is the orthogonal projection on the tangent space to G of the natural affine covariant derivative $\nabla_X Y$ for the affine connection on the ambient linear space.

A straightforward calculation shows that the curve $\theta \mapsto R_\theta$ is orthogonal to the A-orbits, hence, the unit normal vector to AR_θ at R_θ is:

$$n(\theta) = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}$$

Moreover, this unit normal vector is future oriented if we consider the orientation of U for which θ increases with time. Now, for any p, q, consider the curve $t \mapsto c(t) = g^{pt} n(\theta) g^{-qt}$. Its tangent vector at $t = 0$ is:

$$X_{p,q} = \begin{pmatrix} (p-q)\cos\theta & (q+p)\sin\theta \\ (q+p)\sin\theta & (q-p)\cos\theta \end{pmatrix}$$

The unit normal vector $n(t)$ to the A-orbit at $c(t) = g^{pt}R_\theta g^{-qt}$ is

$$g^{pt}n(\theta)g^{-qt} = \begin{pmatrix} -e^{t(p-q)}\sin\theta & e^{t(q+p)}\cos\theta \\ -e^{-t(q+p)}\cos\theta & -e^{t(q-p)}\sin\theta \end{pmatrix}$$

Hence, the derivative at $t = 0$ is:

$$\begin{pmatrix} (q-p)\sin\theta & (q+p)\cos\theta \\ (q+p)\cos\theta & (p-q)\sin\theta \end{pmatrix}$$

The orthogonal projection of this tangent vector to AR_θ at R_θ is the covariant derivative of the unit normal vector along the curve $t \mapsto c(t)$. It follows that the second fundamental form is:

$$II(X_{p,q}, X_{p,q}) = -\langle X_{p,q} \mid \overline{\nabla}_{X_{p,q}} n(t)\rangle = ((p-q)^2 - (p+q)^2)\sin(2\theta)$$

Whereas the first fundamental form, i.e., the metric itself, is:

$$\langle X_{p,q} \mid X_{p,q}\rangle = (p-q)^2\cos^2\theta + (p+q)^2\sin^2\theta$$

Therefore, the principal eigenvalues are $-2\text{cotan}\theta$ and $2\tan\theta$. It follows that the mean curvature value is $\kappa(\theta) = -4\text{cotan}(2\theta)$. The function $\kappa \circ \bar{\theta}$ is then increasing with time: this is the required CMC time function. □

Remark 7.6. *The closure of the domain U meets the conformal boundary at infinity $\partial\mathbb{A}d\mathbb{S}_3$ on a topological nontimelike circle, but it is not a spacelike curve. Actually, the intersection of the closure of U with $\partial\mathbb{A}d\mathbb{S}_3$ is the union of four lightlike geodesic segments (see figure 3).*

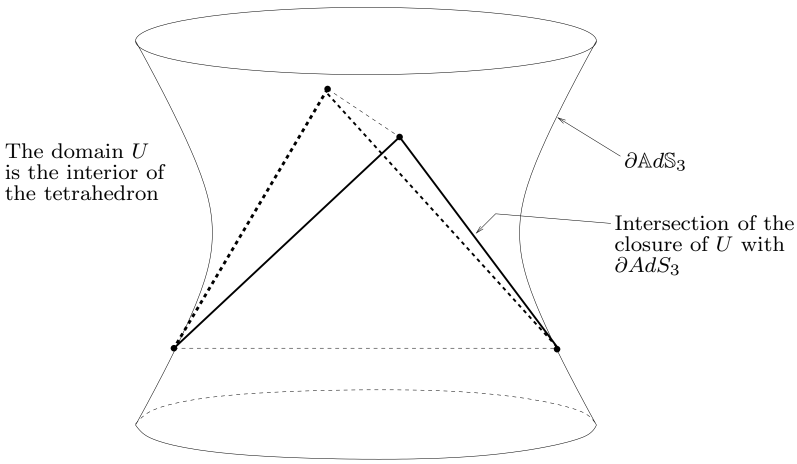

Figure 3: The domain U represented in the projective model of $\mathbb{A}d\mathbb{S}_3$ (more precisely, here we use a projective chart mapping some domain of $\mathbb{A}d\mathbb{S}_3$ in $\mathbb{R}^3$).

Remark 7.7. *The Torus Universes as defined above are the same as those described in [8] in the case of negative cosmological constant (this follows immediatly from the results of subsection 7.2 below). Observe that the expression of the metric on the A-orbit enables to recover easily the features discussed in [8]: the volume of the slices $\bar{\theta} = Cte$ are proportionnal to $\sin 2\theta$, and the conformal classes of these toroidal metrics describe geodesics in the modular space $Mod(T)$ of the torus. More precisely: on the slice $\bar{\theta} = Cte$, the conformal class and the second differential form define naturally a point in the cotangent bundle of $Mod(T)$, and when the Cte is evolving, these data describe an orbit of the geodesic flow on $T^*Mod(T)$. Conversely, every orbit of the geodesic flow on $T^*Mod(T)$ corresponds to a Torus Universe.*

7.2 Every maximal globally hyperbolic spacetime, locally modelled on $\mathbb{A}d\mathbb{S}_3$, with closed Cauchy surfaces of genus 1 is a Torus Universe

In this section, we consider a maximal globally hyperbolic Lorentzian manifold M, locally modelled on AdS_3, whose Cauchy surfaces are 2-tori. We will prove that such a spacetime M is isometric to a Torus Universe (as defined in subsection 7.1). Together with Theorem 7.2, this will imply that M admits a CMC time function.

As in section 6, we consider a Cauchy surface Σ_0 in M, and the lift $\widetilde{\Sigma}_0$ of Σ_0 in the universal covering $\widetilde{M}$ of M. We have a locally isometric developping map $\mathcal{D} : \widetilde{M} \to AdS_3$, and a holonomy representation ρ of $\pi_1(M) = \pi_1(\Sigma_0)$ in the isometry group of AdS_3. We denote $\Gamma = \rho(\pi_1(M)) \subset SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ (here, we prefer to see the isometry group of $\mathbb{A}d\mathbb{S}_3$ as $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ rather than $O(2,2)$), and we denote $S_0 = \mathcal{D}(\widetilde{\Sigma}_0)$. According to Proposition 5.1, S_0 is properly embedded in AdS_3.

The surface σ_0 is a two-torus : hence, the fundamental group of Σ_0 is isomorphic to $\mathbb{Z}^2$. Moreover, according to Proposition 5.1, $\Gamma = \rho(\pi_1(M)$ is a discrete subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Hence, Γ is a lattice in some abelian group $A = H_L \times H_R$, where $H_L = \{e^{th_L}\}_{t\in\mathbb{R}}$ (resp. $H_R = \{e^{sh_R}\}_{s\in\mathbb{R}}$ is a one parameter subgroup of $SL(2,\mathbb{R}) \times \{id\}$ (resp. $\{id\} \times SL(2,\mathbb{R})$). Since A is isomorphic to $\mathbb{R}^2$, these one-parameter groups are either parabolic or hyperbolic. In other words, up to factor switching and conjugacy, there are only three cases to consider:
- *Hyperbolic - hyperbolic*:

$$h_L = h_R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- *Parabolic - parabolic*:

$$h_L = h_R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- *Hyperbolic - parabolic*:

$$h_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad h_R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Let us consider an orbit O of A. The restriction to O of the ambient Lorentzian metric defines a field of quadratic forms on O. Since A is a group of isometries, the quadratic forms appearing in this field have a well-defined type: each of them is either positive definite, negative definite, Lorentzian, or degenerate. We call such a field of quadratic forms a generalized metric. The following lemma describes all the "isometry" type of generalized metrics which can arise by this construction:

Lemma 7.8. *Every orbit O of A has dimension 1 or 2. Moreover:*
- If O has dimension 1, then it is isometric to a line, or to an isotropic line (i.e. equipped with the trivial null generalized metric).
- If O has dimension 2, then it is isometric to the Euclidean plane, the Minkowski plane, or the degenerate plane, i.e. the plane with coordinates (x,y) equipped with the quadratic form dx^2.

Proof. If an element (e^{th_L}, e^{-sh_R}) fixes a point g in $SL(2,\mathbb{R})$, then $e^{th_L} = ge^{sh_R}g^{-1}$. Observe that in the hyperbolic-parabolic case, this implies $s = t = 0$: in this case, every orbit of A is a 2-dimensional plane. In the hyperbolic-hyperbolic case or the parabolic-parabolic case, this implies $s = t$ and $g = e^{th_L}$: hence, there is no 0-dimensional orbit, 1-dimensional orbits are lines, and 2-dimensional orbits are planes.

We parametrize the A-orbit O of an element g_0 of $AdS_3 \approx SL(2,\mathbf{R})$ by $(s,t) \mapsto e^{th_L}g_0e^{-sh_R}$. The differential of this parametrization is:

$$(h_Le^{th_L}g_0e^{-sh_R})ds - (e^{th_L}g_0e^{-sh_R}h_R)dt$$

Since h_R and h_L commute respectively with their exponential, and since these exponentials have determinant 1, the determinant of this expression reduces to the determinant of:

$$(h_L g_0)ds - (g_0 h_R)dt$$

The quadratic form induced on the tangent space of O at (s,t) is $-\det$ of this expression.

If O has dimension 1, then $g_0 h_R g_0^{-1} = h_L = h_R$, thus this determinant is equal to the determinant of $h_L ds - h_L dt$. In the parabolic-parabolic case, we obtain identically 0: O is an isotropic line. In the hyperbolic-hyperbolic case, we obtain $(d(s-t))^2$: O is a Euclidean line.

When O has dimension 2, it is diffeomorphic to the plane. Observe that in the expression above, s and t appear only by their differentials: this means that the generalized metric is actually a parallel field of quadratic forms. In other words, it is given by the quadratic form $-\det(h_L g_0 ds - g_0 h_R dt)$ on the 2-plane O with linear coordinates (s,t). The lemma follows from the classification of quadratic forms on the plane (the negative definite case and the case $-(dx)^2$ are excluded since the quadratic form is obtained by the restriction of a Lorentzian quadratic form). □

Lemma 7.9. *The surface S_0 intersects only 2-dimensional spacelike orbits of A.*

Proof. Let O be the A-orbit of an element x_0 of S_0. Assume first that O has dimension 1: according to Lemma 7.8, O is a line. Observe that O is preserved by the action of Γ. Since Γ acts freely on S_0, x_0 is not fixed by any element of Γ. Hence, every Γ-orbit in O is dense. It follows that there are Γ-iterates of x_0 arbitrarly close to x_0. This is impossible, since Γ acts properly in a neighbourhood of S_0.

Therefore, O has dimension 2. Assume that O is not spacelike. According to Lemma 7.8, it is isometric to the Minkowski plane or the degenerate plane. Since S_0 is spacelike, S_0 and O are transverse. Their intersection is a closed 1-manifold L. Moreover, the ambient Lorentzian metric restricts as a metric on L which is complete. The argument used in Proposition 5.1 can then be applied once more: if O is a Minkowski plane, L intersects every timelike line in O in one and only one point, and if O is degenerate, the same argument proves that L must intersect every degenerate line $y = Cte$ in one and only one point (in this situation, the projection of L on the coordinate x is an isometry!).

It follows that in both cases, L is connected. Therefore, it is isometric to $\mathbb{R}$. But since O and S_0 are both preserved by Γ, the same is true for L: we obtain that $L \approx \mathbb{R}$ admits a free and properly discontinuous isometric action by $\Gamma \approx \mathbb{Z}^2$. Contradiction. □

According to the lemma, some orbits of A are spacelike, and this excludes all the cases except the hyperbolic-hyperbolic case. Hence, A is precisely the abelian group of isometries studied in subsection 7.1 for the definition of the Torus Universes. Moreover, Lemma 7.9 states precisely that S_0 is contained in a connected component U of the domain Ω. Since this is true for any Cauchy surface Σ, and since M is globally hyperbolic, the image of the developping map is contained in U. Hence, M embedds isometrically in the Torus Universe $\Gamma\backslash U$. Since M is maximal as a globally hyperbolic spacetime, M is actually isometric to this quotient.

Thus, we have proved:

Theorem 7.10. *Every maximal globally hyperbolic Lorentzian manifold, locally modelled on AdS_3, with closed oriented Cauchy surfaces of genus 1 is isometric to a Torus Universe.*

Corollary 7.11. *Torus Universes are maximal as globally hyperbolic spacetimes.*

Proof of Theorem 1.1 in the case $g = 1$. The result follows from Theorem 7.10 and 7.2. □

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References

[1] L. Andersson. Constant mean curvature foliations in flat spacetimes. *Comm. Anal. Geom.* **10** (2002), no. 5, 1125-1150.

[2] L. Andersson, G. Galloway and R. Howard. The cosmological time function. *Classical Quantum Gravity* **15** (1998), no. 2, 309–322.

[3] L. Andersson and V. Moncrief. Elliptic-hyperbolic systems and the Einstein equations. *Ann. Henri Poincaré* **4** (2003), no. 1, 1–34.

[4] L. Andersson, V. Moncrief, A. J. Tromba. On the global evolution problem in $2+1$ gravity. *J. Geom. Phys.* **23** (1997), no. 3-4, 191–205.

[5] T. Barbot. Flat globally hyperbolic spacetimes. To appear in *Journal of Geometry and Physics.*

[6] T. Barbot, F. Béguin et A. Zeghib. Feuilletage des espaces-temps globalement hyperboliques par des hypersurfaces à courbure moyenne constante. *C. R. Acad. Sci. Paris*, **336**, 3 (2003), pages 245–250.

[7] T. Barbot and A. Zeghib. Group actions on Lorentz spaces, Mathematical aspects: a survey, in *50 years of the Cauchy problem*, P. Chrusciel and H. Friedrich ed., Birkhaüser, 2004.

[8] S. Carlip. *Quantum gravity in* $2+1$ *dimensions.* Cambridge Monographs on Math. Phys., Cambridge University Press, 1998.

[9] Y. Fourès-Bruhat, Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Mathematica* **88** (1952), 141-225.

[10] Y. Choquet-Bruhat and R. Geroch. Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.* **14** (1969), 329–335.

[11] C. Gerhardt. H-surfaces in Lorentzian manifolds. *Comm. Math. Phys.* **89** (1983), no 4, 523-533.

[12] W. Goldman. Topological components of spaces of representations. *Inv. Math.* **93** (1988), 557-607.

[13] W. Goldman. Geometric structures on manifolds and varieties of representations. *Geometry of group representations* (Boulder, CO, 1987), 169–198, Contemp. Math., 74, Amer. Math. Soc., Providence, RI, 1988.

[14] G. Ellis, S. W. Hawking. *The large scale structure of spacetime.* Cambridge Monographs on Mathematical Physics, Cambridge university Press, 1973.

[15] G. Mess. Lorentz spacetimes of constant curvature. Preprint IHES/M/90/28 (1990).

[16] V. Moncrief. Reduction of the Einstein equations in $2+1$ dimensions to a Hamiltonian system over Teichmüller space. *J. Math. Phys.* **30** (1989), no. 12, 2907–2914.